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BY

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UN. A.

PREFACE

The Author's Complete School Algebra was written to meet the wants of our Common and High Schools and Academies, and to afford adequate preparation for entering our best Colleges, Schools of Science, and Universities.

The present volume is designed for use in these advanced courses of training. Thus, while it is thought that the former affords as extended a course in Algebra as is expedient for the preparatory schools, it is believed that this will be found to contain all that these higher schools require.

It was deemed necessary to make the work a complete treatise, including the Elements, for purposes of reference, and for reviews, and also in consideration of the fact that our higher institutions have various standards of requirement for admission. In fact, there are few students of Higher Algebra who do not find it necessary to have the Elements at hand for occasional consultation.

This Elementary portion is embraced in the first 150 pages, and contains all the definitions, principles, rules, and demonstrations of the Complete School Algebra, with an abundant collection of *New Examples;* but from it all elementary illustrations, explanations, solutions, and suggestions, are omitted. The whole is so arranged as to secure readiness of reference and convenience of review by somewhat mature students.

The subjects treated in Part III., which constitutes the *Advanced Course* proper, will be best seen by turning to the Table of Contents. In this place the author wishes merely to call attention to a few of the distinguishing features of this Part.

1. The conception of *Function* and *Variable* is introduced at once, and is made familiar by such use of it as mathematicians are constantly making. No one needs to be told that this conception lies at the foundation of all higher algebraic discussion ; yet, strangely enough, the very terms are scarcely to be found in our common text-books, and the practical use of the conception is totally wanting.

2. The first chapter in the Advanced Course is given to an elementary and practical exposition of the *Infinitesimal Analysis*. The author knows from his own experience, and from that of many others, that this subject presents no peculiar difficulties to ordinary minds; and everybody knows that it is only by this analysis that the development of functions, as in the Binomial Formula, Logarithmic Series, etc., the general relation of function and variable, the evolution of many of the principles requisite in solving the Higher Equations, and many other subjects, are ever treated by mathematicians, except when they attempt to make Algebras. No mathematician thinks of using the clumsy and antiquated processes by which we have been accustomed to teach our pupils in algebra to demonstrate the Binomial Formula, produce the Logarithmic Series, deduce the law of derived polynomials, examine the relative rate of change of a function and its variable, etc., except when he is teaching the tyro. Why not, then, dismiss forever these processes, and let the pupil enter at once upon those elegant and productive methods of thinking which he will ever after use?

3. By the introduction of a short chapter on *Loci of Equations*, which any one can read even without a knowledge of Elementary Geometry, and which in itself is always interesting to the pupil, and of fundamental use in the subsequent course, *all the more abstruse principles of the Theory of Equations* are illustrated, and the student is thus enabled to *see* the truth, as well as to demonstrate it abstractly. How great an advantage this is, no experienced teacher needs to be told.

4. In the treatment of the Higher Equations, while some things have been discarded which everybody knows to be worthless, but which have in some way found a place in our text-books, a far more full and clear discussion of practical principles and methods is given, than is found in any of the treatises in common use.

5. The important but difficult subject of the *Discussion of Equations* has been reserved till late in the course, for several reasons. Thus, when the pupil reaches this topic, he has become familiar with most of the principles to be applied, and has become sufficiently imbued with the spirit of the algebraic analysis to be enabled to grasp it. To discuss an equation independently and well, is a high mathematical accomplishment, and should not be expected of the tyro. It is nothing else than to think in mathematical formulæ, and hence is one of the later products of mathematical study. It is hoped that the position assigned to this subject in the course, and the manner of treating it, will insure better results than we have hitherto been able to obtain.

6. In the selection of *Subjects to be Presented*, constant regard has been had

to the demands of the subsequent mathematical course. This has led to the omission of a number of theorems and methods, which, though well enough in themselves as mere matter of theory, find no practical application in a subsequent course, however extended; and has, at the same time, led to the introduction of not a few things which the advanced student always finds occasion to use, but for which he searches his Algebra in vain, if he has at hand nothing but our common American text-books.

7. In *Method of Treatment* the following principles have been kept constantly in mind: 1. That the view presented be in line with the mathematical thinking of to-day. 2. That everything be rigidly demonstrated and amply and clearly illustrated. 3. When long experience has shown that the majority of good students have difficulty in comprehending a subject, special pains should be taken to elucidate it. 4. No principle is thoroughly learned by a pupil until he can apply it; and nothing so *fixes* principles in the mind as the use of them. Hence an unusually large number of examples has been introduced. 5. It is often necessary to multiply examples in order to meet the requirements of the class-room.

8. *Answers.*—The answers to examples are not generally annexed to them in the text. There are, however, two editions of the volume, one with the answers at the end, and the other without any answers, except an occasional one in the body of the book.

9. Finally, the *Order of Topics* is such that a student requiring a less extended course than the entire volume presents, can stop at any point, and feel assured that what he has studied is of more elementary importance than what follows. Thus students who do not desire to study the Higher Equations can conclude their course with the first chapter of *Part III.;* and a course which includes the first three chapters of this part will be found as extended as most of our Academies, and perhaps many of our Colleges, will find expedient.

Such works as those of Serret, Cirodde, Comberousse, Wood, Hymers, Hind, Todhunter, Young, and most of our American treatises, have been at hand during the preparation of the entire volume. To Whitworth's charming little treatise on *Choice* and *Chance*, the author is indebted for a number of examples in the last section.

The quick eye and cultivated taste of my friend, Mr. W. W. Beman, A.M., Instructor of Mathematics in the University, have done me excellent service in reading the proof-sheets, and have, I trust, given the work a degree of typographical accuracy not usually found in first issues of such treatises.

With these words of explanation as to what I have attempted to do, I commit the volume to the hands of my fellow-laborers in the work of teaching, assured from the generous and appreciative reception which they have given my previous efforts, that this will not fail of a candid consideration.

EDWARD OLNEY.

UNIVERSITY OF MICHIGAN,
Ann Arbor, July, 1873.

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SECTION I.

GENERAL DEFINITIONS, AND THE ALGEBRAIC NOTATION.

BRANCHES OF PURE MATHEMATICS.

1. ***Pure Mathematics*** is a general term applied to several branches of science, which have for their object the investigation of the properties and relations of quantity—comprehending number, and magnitude as the result of extension—and of form.

2. ***The Several Branches*** of Pure Mathematics are Arithmetic, Algebra, Calculus, and Geometry.

3. Arithmetic, Algebra, and Calculus treat of number, and Geometry treats of magnitude as the result of extension.

4. ***Quantity*** is the amount or extent of that which may be measured; it comprehends number and magnitude.

The term quantity is also conventionally applied to symbols used to represent quantity. Thus 25, m, XI, etc., are called quantities, although, strictly speaking, they are only representatives of quantities.

5. ***Number*** is quantity conceived as made up of parts, and answers to the question, "How many?"

6. Number is of two kinds, ***Discontinuous*** and ***Continuous.***

7. ***Discontinuous Number*** is number conceived as made up of finite parts; or it is number which passes from one state of value to another by the successive additions or subtractions of finite units; *i. e.*, units of appreciable magnitude.

8. ***Continuous Number*** is number which is conceived as composed of infinitesimal parts; or it is number which passes from one state of value to another by passing through all intermediate values, or states.

9. Arithmetic treats of ***Discontinuous Number,***—of its nature and properties, of the various methods of combining and resolving it, and of its application to practical affairs.

10. Algebra treats of the *Equation,* and is chiefly occupied in explaining its nature and the methods of transforming and reducing it, and in exhibiting the manner of using it as an instrument for mathematical investigation.*

11. Calculus treats of *Continuous Number,* and is chiefly occupied in deducing the relations of the infinitesimal elements of such number from given relations between finite values, and the converse process, and also in pointing out the nature of such infinitesimals and the method of using them in mathematical investigation.

12. Geometry treats of *magnitude* and *form* as the result of extension and position.

LOGICO-MATHEMATICAL TERMS.

13. A Proposition is a statement of something to be considered or done.

14. Propositions are distinguished as *Axioms, Theorems, Lemmas, Corollaries, Postulates,* and *Problems.*

15. An Axiom is a proposition which states a principle that is so simple, elementary, and evident as to require no proof.

16. A Theorem is a proposition which states a real or supposed fact, whose truth or falsity we are to determine by reasoning.

17. A Demonstration is the course of reasoning by means of which the truth or falsity of a theorem is made to appear. The term is also applied to a logical statement of the reasons for the processes of a rule. A solution tells *how* a thing is done; a demonstration tells *why* it is so done. A demonstration is often called *proof.*

---

* The common definition of Algebra, which makes its distinguishing features to be *the literal notation*, and *the use of the signs*, is entirely at fault. When Algebra first appeared in Europe, it possessed neither of these features! What was it then? On the other hand, the signs are common to all branches of mathematics, and the literal notation is as prominent in the Calculus as in Algebra, and is used, more or less, in common Arithmetic and Geometry.

18. *A Lemma* is a theorem demonstrated for the purpose of using it in the demonstration of another theorem.

19. *A Corollary* is a subordinate theorem which is suggested, or the truth of which is made evident, in the course of the demonstration of a more general theorem, or which is a direct inference from a proposition.

20. *A Postulate* is a proposition which states that something can be done, and which is so evidently true as to require no process of reasoning to show that it is possible to be done. We may or may not know how to perform the operation.

21. *A Problem* is a proposition to do some specified thing, and is stated with reference to developing the method of doing it.

22. *A Rule* is a formal statement of the method of solving a general problem, and is designed for practical application in solving special examples of the same class. Of course a rule requires a demonstration.

23. *A Solution* is the process of performing a problem or an example. It should usually be accompanied by a demonstration of the process.

24. *A Scholium* is a remark made at the close of a discussion, and designed to call attention to some particular feature or features of it.

PART I.*

LITERAL ARITHMETIC.†

CHAPTER I.

FUNDAMENTAL RULES.

SECTION I.

NOTATION.

25. ***A System of Notation*** is a system of symbols by means of which quantities, the relations between them, and the operations to be performed upon them, can be more concisely expressed than by the use of words.

SYMBOLS OF QUANTITY.

26. In Arithmetic, as usually studied, numbers are represented by the characters, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, called Arabic figures, or, simply, figures.

27. In other departments of mathematics than Arithmetic, numbers or quantities are more frequently represented by the common letters of the alphabet, $a, b, c, \ldots m, n, \ldots x, y, z$. These letters may, however, be used in Arithmetic; and the Arabic figures are used in all departments of mathematics. This method of represent-

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* PARTS I. and II. are a compend of the elements of the science, designed as a review for pupils who have studied some elementary treatise, or for the use of such teachers and classes as desire a text-book which contains a condensed treatment of the subject, to be filled out by themselves. In the author's COMPLETE SCHOOL ALGEBRA, the topics here presented will be found fully amplified, illustrated, and applied. All the elementary principles are here stated, and are usually demonstrated. There are also numerous examples under every topic. The KEY to the COMPLETE SCHOOL ALGEBRA will furnish additional examples for use in connection with this part.

† Part I. treats of the familiar operations of Addition, Subtraction, Multiplication, Division, Involution and Evolution, and the theory of Fractions. The only difference between the processes here developed and the corresponding ones in *common* Arithmetic grows out of the notation.

ing quantities by letters is often called the *Algebraic* method, and the method by the Arabic characters the *Arithmetical.* It would be better to call the former the *Literal* method, and the latter the *Decimal.*

28. The Literal Notation has some very great advantages over the decimal for purposes of mathematical reasoning. 1st, The symbols are more general in their signification; and 2d, We are enabled to detect the same quantity anywhere in the process, and even in the result. Thus it happens that the processes become general *formulæ,* or rules, instead of special solutions.

29. In using the decimal notation certain *laws* are established, in accordance with which all numbers can be represented by the ten figures. Thus, it is agreed that when several figures stand together without any other mark, as 435, the right-hand figure shall signify units, the second to the left, tens, the third, hundreds, etc.; also that the *sum* of the several values shall be taken. This number is, therefore, 4 hundreds + 3 tens + 5 (units).

In like manner, certain laws are observed in representing numbers by letters.

FIRST LAW.

30. Known Quantities, that is such as are given in a problem, are represented by letters taken from the first part of the alphabet; while **Unknown Quantities**, or quantities whose values are to be found, are represented by letters taken from the latter part of the alphabet.

Accented letters, as a', a'', a''', a'''', etc., (read "*a* prime," "*a* second," "*a* third," etc.,) and letters with subscripts, as a_1, a_2, a_3, a_4, etc., (read "*a* sub 1," "*a* sub 2," etc.,) are sometimes used. This form of notation is used when there are several *like* quantities in the same problem, but which have different numerical values. Thus, in a problem in which several walls of different heights, breadths, and lengths are considered, we may represent the several heights by a', a'', a''', etc., or a_1, a_2, a_3, etc.; the thicknesses by b', b'', b''', etc., or b_1, b_2, b_3, etc., and the lengths by l', l'', l''', etc., or l_1, l_2, l_3, etc.

The Greek letters are also often used both for known and unknown quantities.

SECOND LAW.

31. When letters are written in connection, without any sign between them, their product is signified. Thus *abc* signifies that the three numbers represented by *a*, *b*, and *c* are to be multiplied together.

32. A character like a figure 8 placed horizontally, ∞, is used to represent what is called *Infinity*, or a quantity larger than any assignable quantity.

Symbols of Operation.

33. The Symbols of Operation used in Algebra are the same as those used in Arithmetic, or in any other branch of mathematics, and need not be recapitulated here.

Exponents.

34. An Exponent is a small figure, letter, or other symbol of number, written at the right and a little above another figure, letter, or symbol of number.*

35. A Positive Integral Exponent signifies that the number affected by it is to be taken as a factor as many times as there are units in the exponent. It is a kind of symbol of multiplication.

36. A Positive Fractional Exponent indicates a power of a root, or a root of a power. The denominator specifies the root, and the numerator the power of the number to which the exponent is attached.

37. The Radical Sign, $\surd$, is also used to indicate the square root of a quantity. When any other than the square root is to be designated by this, a small figure specifying the root is placed in the sign.

38. A Negative Exponent, *i. e.*, one with the $-$ sign before it, either integral or fractional, signifies the reciprocal of what the expression would be if the exponent were positive, *i. e.*, had the $+$ sign, or no sign at all before it.

Symbols of Relation.

39. The Sign of Geometrical Ratio is two dots in the form of a colon, : .

40. The Sign of Arithmetical Ratio is two dots placed horizontally, ·· .

41. The Sign of Equality is two parallel horizontal lines, $=$. The double colon, : :, is the sign of equality between ratios.

---

* In giving this definition, be careful and *not* add, "and indicates the *power* to which the number is to be raised." This is false : an exponent does not necessarily indicate a power.

42. *The Sign of Variation* is somewhat like a figure 8 open at one end and placed horizontally, $\propto$.

43. *The Sign of Inequality* is a character somewhat like a capital V placed on its side, $<$, the opening being towards the greater quantity.

Symbols of Aggregation.

44. *A Vinculum* is a horizontal line placed over several terms, and indicates that they are to be taken together. The parenthesis, ( ), the brackets, [ ], and the brace, $\{\ \}$, have the same signification.

45. A vertical line after a column of quantities, each having its own sign, signifies that the aggregate of the column is to be taken as one quantity. Thus, $\left.\begin{array}{r} +a \\ -b \\ +c \end{array}\right| x$ is the same as $(a - b + c)x$.

Symbols of Continuation.

46. A series of dots, ..., or of short dashes, - - - - - -, written after a series of expressions, signifies "etc." Thus, $a : ar : ar^2 : ar^3 \ldots\ldots ar^n$ means that the series is to be extended from ar^3 to ar^n, whatever may be the value of n.

Symbols of Deduction.

47. Three dots, two being placed horizontally and the third above and between, $\therefore$, signify *therefore*, or some analogous expression. If the third dot is below the first two, $\because$, the symbol is read "since," "because," or by some equivalent expression.

Positive and Negative Quantities.

48. *Positive and Negative* are terms primarily applied to concrete quantities which are, by the conditions of a problem, opposed in character.

Ill.—A man's *property* may be called positive, and his *debts* negative. Distance *up* may be called positive, and distance *down*, negative. Time *before* a given period may be called positive, and *after*, negative. Degrees *above* 0 on the thermometer scale are called positive, and *below*, negative.

49. The signs + and − are used to indicate the *character* of quantities as positive or negative, as well as for the purpose of indicating addition and subtraction.

50. In problems in which the distinction of positive and negative is made, each quantity in the *formulæ* is to be considered as having a *sign of character* expressed or understood besides the plus or minus sign, which latter indicates that it is to be added or subtracted. The positive sign need not be written to indicate character, as it is customary to consider quantities whose character is not specified as positive.

ILL. 1.—In the expression $ab + m - cx$, let the problem out of which it arose be such, that a, m, and x tend to a positive result, and b and c to an opposite, or a negative result. Giving these quantities their signs of character, we have $(+a) \times (-b) + (+m) - (-c) \times (+x)$, which may be read, "positive a multiplied by negative b, plus positive m, minus negative c multiplied by positive x." Suppressing the positive sign, this may be written, $a(-b) + m - (-c)x$, by also omitting the unnecessary sign of multiplication.

ILL. 2.—As this subject is one of fundamental importance, let careful attention be given to some further illustrations. We are to distinguish between discussions of the relations between mere abstract quantities, and problems in which the quantities have some concrete signification. Thus, if it is desired to ascertain the sum or difference of 468, or m, and 327, or n, as mere numbers, the question is one concerning the relation of abstract numbers, or quantities. No other idea is attached to the expressions than that each represents a certain number of units. But, if we ask how far a man is from his starting point, who has gone, first, 468, or m miles directly east, and then 327, or n miles directly west; or if we ask what is the difference in time between 468, or m years B. C., and 327, or n years A. D., the numbers 468, or m, and 327, or n, take on, besides their primary signification as quantities, the additional thought of *opposition in direction*. They therefore become, in this sense, concrete.

Again, a company of 5 boys are trying to move a wagon. Three of the boys can pull 75, 85, and 100 pounds each; and they exert their strength to move the wagon east. The other two boys can pull 90 and 110 pounds each; and they exert their strength to move the wagon west. It is evident that the 75, 85, and 100 are quantities of an opposite character, in their relation to the problem, from 90 and 110. Again, suppose a party rowing a boat up a river. Their united strength would propel the boat 8 miles per hour if there were no current; but the force of the current is sufficient to carry the boat 2 miles per hour. The 8 and 2 are quantities of opposite character in their relation to the problem. Once more, in examining into a man's business, it is found that he has a farm worth m dollars, personal property worth n dollars, and accounts due him worth c dollars. There is a mortgage on his farm of b dollars, and he owes on account a dollars. The m, n, and c are quantities opposite in their nature to b and a. *This opposition in character is indicated by calling those quantities which contribute to one result positive, and those which contribute to the opposite result negative.*

51. Purely abstract quantities have, properly, no distinction as positive and negative; but, since in such problems the plus or additive, and the minus or subtractive *terms* stand in the same relation to each other as positive and negative quantities, it is customary to call them such.

ILL.—In the expression $5ac - 3cd + 8xy - 2ad$, though the quantities, a, c, d, x and y be merely abstract, and have no proper signs of character of their own, the *terms* do stand in the same relation to each other and to the result, as do positive and negative quantities. Thus, $5ac$ and $8xy$ tend, as we may say, *to increase* the result, while $-3cd$, and $-2ad$ tend *to diminish* it. Therefore the former may be called positive *terms*, and the latter negative.

52. SCH.—*Less than zero.* Negative quantities are frequently spoken of as "less than zero." Though this language is not philosophically correct, it is in such common use, and the thing signified is so sharply defined and easily comprehended, that its use may possibly be allowed as a conventionalism. To illustrate its meaning, suppose, in speaking of a man's pecuniary affairs, it is said that he is worth "less than nothing;" it is simply meant that his debts exceed his assets. If this excess were $1000, it might be called negative $1000, or −$1000. So, again, if a man were attempting to row a boat up a stream, but with all his effort the current bore him down, his progress might be said to be less than nothing, or negative. In short, in any case where quantities are reckoned both ways from zero, if we call those reckoned one way greater than zero, or positive, we may call those reckoned the other way "less than zero," or negative.

53. The value of a Negative Quantity is conceived to *increase* as its numerical value *decreases*.

ILL.—Thus $-3 > -5$, as a man who is in debt $3 is better off than one who is in debt $5, other things being equal. If a man is striving to row up stream, and at first is borne down 5 miles an hour, but by practice comes to row so well as only to be borne down 3 miles an hour, he is evidently gaining; *i. e.*, -3 is an *increase* upon -5. Finally, consider the thermometer scale. If the mercury stands at 20° below 0 (marked $-20°$) at one hour, and at $-10°$ the next hour, the temperature is increasing; and, if it increase sufficiently, will become 0, *passing which* it will reach $+1°$, $+2°$, etc. *In this illustration, the quantity passes from negative to positive by passing through* 0.

It appears in geometry, that a quantity may also change its sign in passing through *infinity*. Thus the tangent of an arc less than 90° is positive; but if the arc continually increases, the tangent becomes infinity at 90°, passing which it becomes negative.

Now, as we know of no other way in which a varying quantity can change its sign, it is assumed as a fundamental principle in mathematics that, IF A VARYING QUANTITY CHANGES ITS SIGN, IT PASSES THROUGH ZERO, OR INFINITY.

NAMES OF DIFFERENT FORMS OF EXPRESSION.

54. ***A Polynomial*** is an expression composed of two or more parts connected by the signs plus and minus, each of which parts is called a *term.*

55. *A Monomial* is an expression consisting of one term; a ***Binomial*** has two terms; a ***Trinomial*** has three terms, etc.

56. *A Coefficient* of a term is that factor which is considered as denoting the number of times the remainder of the term is taken. The numerical factor, or the product of the known factors in a term, is most commonly called the coefficient, though any factor, or the product of any number of factors in a term may be considered as coefficient to the other part of the term.

57. *Similar Terms* are such as consist of the same letters affected with the same exponents.

SECTION II.

ADDITION.

58. *Addition* is the process of combining several quantities, so that the result shall express the aggregate value in the fewest terms consistent with the notation.

59. *The Sum or Amount* is the aggregate value of several quantities, expressed in the fewest terms consistent with the notation.

60. *Prop. 1.* *Similar terms are united by Addition into one.*

DEM.—Let it be required to add $4ac$, $5ac$, $-2ac$, and $-3ac$. Now $4ac$ is 4 times ac, and $5ac$ is 5 times the same quantity (ac). But 4 times and 5 times the same quantity make 9 times that quantity. Hence, $4ac$ added to $5ac$ make $9ac$. To add $-2ac$ to $9ac$ we have to consider that the negative quantity, $-2ac$, is so opposed in its character to the positive, $9ac$, as to tend to destroy it when combined (added) with it. Therefore, $-2ac$ destroys 2 of the 9 times ac, and gives, when added to it, $7ac$. In like manner, $-3ac$ added to $7ac$, gives $4ac$. Thus the four similar terms, $4ac$, $5ac$, $-2ac$, and $-3ac$, have been combined (added) into one term, $4ac$; and it is evident that any other group of similar terms can be treated in the same manner. Q. E. D.

61. COR. 1.—*In adding similar terms, if the terms are all positive, the sum is positive; if all negative, the sum is negative; if some are positive and some negative, the sum takes the sign of that kind (positive or negative) which is in excess.*

SCH.—The operation of adding positive and negative quantities may look to the pupil like Subtraction. For example, we say $+5$ and -3 added make

+2. This looks like Subtraction, and, in one view, it is Subtraction. But why call it Addition? The reason is, because it is simply *putting the quantities together*—aggregating them—not *finding their difference.* Thus, if one boy pulls on his sleigh 5 pounds in one direction, while another boy pulls 3 pounds in the *opposite* direction, the combined (added) effect is 2 pounds in the direction in which the first pulls. If we call the *direction* in which the first pulls positive, and the *opposite* direction negative, we have +5 and −3 to add. This gives, as illustrated, +2. Hence we see, that the sum of +5 and −3 is +2.

But the *difference* of + 5 and − 3 is 8, as will appear from the following illustration: Suppose one boy is trying to draw a sleigh in a certain direction, and another is holding back 3 lbs. If it takes 10 lbs. to move the sleigh, the first boy will have to pull 13 lbs. to get it on. But if, instead of *holding back* 3 lbs., the second boy *pushes* 5 lbs., the first boy will have to pull only 5 lbs. Thus it appears, that the *difference* between pushing 5 lbs. (or + 5) and *holding* back 3 lbs. (−3) is 8 lbs.

In like manner the sum of $25 of property and $15 of debt, that is the aggregate value when they are combined, is $10. +25 and −15 are +10. But the *difference* between having $25 in pocket, and being $15 in debt, is $40. The difference between +25 and −15 is 40.

62. Cor. 2.—*The sum of two quantities, the one positive and the other negative, is the numerical difference, with the sign of the greater prefixed.*

63. Cor. 3.—*It appears that addition in mathematics does not always imply increase. Whether a quantity is increased or diminished by adding another to it, depends upon the relative nature of the two quantities. If they both tend to the same end, the result is an increase in that direction. If they tend to opposite ends, the result is a diminution of the greater by the less.*

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64. Prop. 2. *Dissimilar terms are not united into one by addition, but the operation of adding is expressed by writing them in succession, with the positive terms preceded by the + sign, and the negative by the − sign.*

Dem.—Let it be required to add $+4cy^2$, $+3ab$, $-2xy$, and $-mn$. $4cy^2$ is 4 times cy^2, and $3ab$ is 3 times ab, a different quantity from cy^2; the sum will, therefore, not be 7 times, nor, so far as we can tell, any number of times cy^2 or ab, or any other quantity, and we can only *express* the addition thus: $4cy^2+3ab$. In like manner, to add to this sum $-2xy$ we can only express the addition, as $4cy^2+3ab+(-2xy)$. But since $2xy$ is negative, it tends to destroy the positive quantities and will take out of them $2xy$. Hence the result will be $4cy^2+3ab-2xy$. The effect of $-mn$ will be the same in kind as that of $-2xy$, and hence the total sum will be $4cy^2+3ab-2xy-mn$. As a similar course of reasoning can be applied to any case, the truth of the proposition appears.

SCH.—In such an expression as $4cy^2 + 3ab - 2xy - mn$, the $-$ sign before the mn does not signify that it is to be taken from the immediately preceding quantity; nor is this the signification of any of the signs. But the quantities having the $-$ sign are considered as operating to destroy *any* which may have the $+$ sign, and *vice versa*.

65. COR.—*Adding a negative quantity is the same as subtracting a numerically equal positive quantity; that is,* $m + (-n)$ *is* $m - n$, *shown as above.*

DEM.—Since a negative quantity is one which tends to destroy a positive quantity, $-n$ when added to m (*i.e.* $+m$) destroys n of the units in m, and hence gives as a result $m - n$.

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66. Prob.—*To add polynomials.*

RULE.—COMBINE EACH SET OF SIMILAR TERMS INTO ONE TERM, AND CONNECT THE RESULTS WITH THEIR OWN SIGNS. THE POLYNOMIAL THUS FOUND IS THE SUM SOUGHT.*

DEM.—The purpose of addition being to combine the quantities so as to express the aggregate (sum) in the fewest terms consistent with the notation, the correctness of the rule is evident, as *only* similar terms can be united into one (60, 64).

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67. Prop. 3. *Literal terms, which are similar only with respect to part of their factors, may be united into one term with a polynomial coefficient.*

DEM.—Let it be required to add $5ax$, $-2cx$, and $2mx$. These terms are similar, only with respect to x, and we may say $5a$ times x and $-2c$ times x make $(5a - 2c)$ times x, or $(5a - 2c)x$. And then, $5a - 2c$ times x and $2m$ times x make $(5a - 2c + 2m)$ times x, or $(5a - 2c + 2m)x$. Q. E. D.

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68. Prop. 4. *Compound terms which have a common compound, or polynomial factor, may be regarded as similar and added with respect to that factor.*

DEM. $5(x^2 - y^2)$, $2(x^2 - y^2)$ and $-3(x^2 - y^2)$ make, when added with respect to $(x^2 - y^2)$, $4(x^2 - y^2)$, for they are $5 + 2 - 3$, or 4 times the same quantity $(x^2 - y^2)$. In a similar manner we may reason on other cases. Q. E. D.

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* This is the proficient's rule, as exhibited on page 45 of the COMPLETE SCHOOL ALGEBRA, SCH. 2.

SCH.—The object and process of addition, as now explained, will be seen to be identical with the same as the pupil has learned them in Arithmetic, except what grows out of the notation, and the consideration of positive and negative quantities. For example, in the decimal notation let it be required to add 248, 10506, 5003, 81, and 106. The units in the several numbers are similar terms, and hence are combined into one: so also of the tens, and of the hundreds. The process of carrying has no analogy in the literal notation, since the relative values of the terms are not supposed to be known. Again, there is nothing usually found in the decimal addition like positive and negative quantities. With these two exceptions the processes are essentially the same. The same may be said of addition of compound numbers.

EXAMPLES.

1. Find the sum of $2a - 3x^2$, $5x^2 - 7a$, $-3a + x^2$, and $a - 3x^2$.

2. Find the sum of $a^2 - b^2 + 3a^2b - 5ab^2$, $3a^2 - 4a^2b + 3b^3 - 3ab^2$, $a^3 + b^3 + 3a^2b$, $2a^3 - 4b^3 - 5ab^2$, $6a^2b + 10ab^2$, and $-6a^3 - 7a^2b + 4ab^2 + 2b^3$.

3. Find the sum of $5ca^2x^2 + 4ba^2x^2 + mx^2y^2$, and $10ca^2x^2 - 2ba^2x^2 + 6mx^2y^2$.

4. Add $2x^{\frac{1}{2}} - 4x^{\frac{1}{3}} + x^2$, $5x^2y - ab + x^{\frac{1}{3}}$, $4x^2 - x^3$, and $2x^{\frac{1}{3}} - 3 + 2x^{\frac{1}{2}}$.

5. Add $\frac{1}{2}(x + y)$ and $\frac{1}{2}(x - y)$.

6. Add $ax+2by+cz$, $\sqrt{x} + \sqrt{y} + \sqrt{z}$, $3y^{\frac{1}{2}}-2x^{\frac{1}{2}} + 3z^{\frac{1}{2}}$, $4cz - 3ax - 2by$, and $2ax - 4\sqrt{y} - 2z^{\frac{1}{2}}$.

7. Add $cz - 2ay$, $2az - 3ay$, $my - az$, with respect to z and y.

8. Add $(a+b)\sqrt{x}-(2+m)\sqrt{y}$, $4y^{\frac{1}{2}} + (a+c)x^{\frac{1}{2}}$, $3n\sqrt{y}-(2d-e)x^{\frac{1}{2}}$, $-2n\sqrt{x} + 12a\sqrt{y}$, and $(m+n)y^{\frac{1}{2}} + (b+2c)\sqrt{x}$.

9. Add $x^2 + xy + y^2$, $ax^2 - axy + ay^2$, and $-by^2 + bxy + bx^2$.

10. Add $a(x + y) + b(x - y)$, $m(x + y) - n(x - y)$.

11. Add $3m\sqrt{x - y} + 6n\sqrt{x - y} - 6\sqrt{x - y} - 3n\sqrt{x - y}$.

12. Add $3ax^{-\frac{1}{2}} + 5y^{-1} - 2c$, $\dfrac{4a}{\sqrt{x}} - \dfrac{2}{y} + 8c$, and $-6ax^{-\frac{1}{2}} - my^{-1} - 3c$.

13. Add $\frac{1}{2}\sqrt[5]{a^2 - x^2}$, $-\frac{2}{3}\sqrt[5]{a^2 - x^2}$, and $\sqrt[5]{a^2 - x^2}$.

14. Add $\frac{a+b-c}{\sqrt{x^2-1}}$, $\frac{a-b+c}{(x^2-1)^{\frac{1}{2}}}$, and $-2a(x^2-1)^{-\frac{1}{2}}$.

15. Add $\frac{1}{2}\left(\sqrt[3]{x^2} - y^{-\frac{1}{3}} + z^{-2}\right)$, and $\frac{2}{3}\left(x^{\frac{2}{3}} + \frac{1}{\sqrt[3]{y}} - \frac{1}{z^2}\right)$.

16. Add $(a-b+c)\sqrt{x^2-y^2}$, $(a+b-c)(x^2-y^2)^{\frac{1}{2}}$, and $(b+c-a)\sqrt{x^2-y^2}$.

17. Condense the polynomial $4ax^{\frac{1}{2}} - 3y^2 + 2cz - 4m\sqrt{x} + 3my^2 - 2ax^{\frac{1}{2}} + 6cz$, into $2(a-2m)\sqrt{x} + 3(m-1)y^2 + 8cz$.

SECTION III.

SUBTRACTION.

69. Subtraction is, primarily, the process of taking a *less* quantity from a greater. In an enlarged sense, it comes to mean taking one quantity from another, irrespective of their magnitudes. It also comprehends all processes of finding the difference between quantities. In all cases the result is to be expressed in the fewest terms consistent with the notation used.

70. The Difference between two quantities is, in its primary signification, the number of units which lie between them; or, *it is what must be added to one in order to produce the other*. When it is required to take one quantity from another, *the difference is what must be added to the Subtrahend in order to produce the Minuend.*

71. Prob.*—To perform Subtraction.*

RULE.—CHANGE THE SIGNS OF EACH TERM IN THE SUBTRAHEND FROM + TO −, OR FROM − TO +, OR CONCEIVE THEM TO BE CHANGED, AND ADD THE RESULT TO THE MINUEND.

DEM.—Since the difference sought is what must be added to the subtrahend to produce the minuend, we may consider this difference as made up of two parts, one the subtrahend with its signs changed, and the other the minuend. When the sum of these two parts is added to the subtrahend, it is evident that the first part will destroy the subtrahend, and the other part, or minuend, will be the sum.

Thus, to perform the example:

From	$5ax - 6b - 3d - 4m$	
Take	$2ax + 2b - 5d + 8m$	If these three quantities are added together, the sum will
Subtrahend with signs changed,	$-2ax - 2b + 5d - 8m$	
Minuend,	$5ax - 6b - 3d - 4m$	
Difference,	$3ax - 8b + 2d - 12m$	

evidently be the minuend. If, therefore, we add the second and third of them (that is, the subtrahend, with its signs changed, and the minuend) together, the sum will be what is necessary to be added to the subtrahend to produce the minuend, and hence is the difference sought. Q. E. D.

72. Cor. 1.—*When a parenthesis, or any symbol of like signification (**44**), occurs in a polynomial, preceded by a − sign, and the parenthesis or equivalent symbol is removed, the signs of all the terms which were within must be changed, since the − sign indicates that the quantity within the parenthesis is a subtrahend.*

73. Cor. 2.—*Any quantity can be placed within a parenthesis, preceded by the − sign, by changing all the signs. The reason of this is evident, since by removing the parenthesis according to the preceding corollary, the expression would return to its original form.*

Examples.

1. How much must be added to 8 to produce 12? What is the difference between 8 and 12? How much must be added to $3ax^2 - 5y^3$ (the subtrahend) to produce $8ax^2 + 2y^3$?

Answer.—To $3ax^2$ we must add $5ax^2$; and to $-5y^3$ we must add $+7y^3$. Hence in all we must add $5ax^2 + 7y^3$.

2. From $3x^3 - 2x^2 - x - 7$ take $2x^3 - 3x^2 + x + 1$.

3. From $a^2 - x^2$ take $a^2 + 2ax + x^2$.

4. From $1 + 3x^{\frac{1}{2}} + 3x + x^{\frac{3}{2}}$ take $1 - 3x^{\frac{1}{2}} + 3x - x^{\frac{3}{2}}$.

5. From $x^{\frac{4}{5}} + 2x^{\frac{2}{5}}y^{\frac{2}{5}} + y^{\frac{4}{5}}$ take $x^{\frac{4}{5}} - 2x^{\frac{2}{5}}y^{\frac{2}{5}} + y^{\frac{4}{5}}$.

6. From $7\sqrt[3]{1 + x^2} - 3ay^{\frac{1}{2}}$ take $-3\sqrt[3]{1 + x^2} + 3ay^{\frac{1}{2}}$.

7. From $ay^2 + 10\sqrt{ab}$ take $ay + x\sqrt{ab}$.

8. From $bx^2 - 3\sqrt{mn} + 1$ take $b^2x + (mn)^{\frac{1}{2}} - 1$.

9. From $a + b + \sqrt{a - b}$ take $b + a - (a - b)^{\frac{1}{2}} + \sqrt{ab}$.

10. Remove the parentheses from the following:

$a - \{(b - c) - d\}$; $7a - \{3a - [4a - (5a - 2a)]\}$;

$2(a - b) - c + d - \{a - b - 2(c - d)\}$;

$3(2a - b - c) - 5\{a - (2b + c)\} + 2\{b - (c - a)\}$.

11. Include within brackets the 3d, 4th, and 5th terms of $3ab - x^2 + ax - 10by + 50$. Also the 4th and 5th. Also the 2d and 3d.

THEORY OF SUBTRACTION.—Subtraction is finding the difference between quantities, that is, finding what must be added to one quantity to produce the other. This difference may always be considered as consisting of two parts, one of which destroys the subtrahend, and the other part is the minuend itself. Hence, to perform subtraction, we change the signs of the subtrahend to get that part of the difference which destroys the subtrahend, and add this result to the minuend, which is the other part of the difference.

SECTION IV.

MULTIPLICATION.

74. *Multiplication* is the process of finding the simplest expression consistent with the notation used, for a quantity which shall be as many times a specified quantity, or such a part of that quantity, as is represented by a specified number.

75. COR. 1.—*The multiplier must always be conceived as an abstract number, since it shows* HOW MANY TIMES *the multiplicand is to be taken.*

76. COR. 2.—*The product is always of the same kind as the multiplicand.*

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77. *Prop. 1.*—*The product of several factors is the same in whatever order they are taken.*

DEM.—1st. $a \times b$, is a taken b times, or $a + a + a + a + a$ - - - - - to b terms. Now, if we take 1 unit from each term (each a), we shall get b units; and this process can be repeated a times, giving a times b, or $b \times a$. $\therefore a \times b = b \times a$.

2d. When there are more than two factors, as abc. We have shown that $ab = ba$. Now call this product m, whence $abc = mc$. But by part 1st, $mc = cm$. $\therefore abc = bac = cab = cba$. In like manner we may show that the product of any number of factors is the same in whatever order they are taken. Q. E. D.

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78. *Prop. 2.*—*When two factors have the same sign their product is positive: when they have different signs their product is negative.*

DEM.—1st. Let the factors be $+ a$ and $+ b$. Considering a as the multiplier we are to take $+ b$, a times, which gives $+ ab$, a being considered as abstract in the operation, and the product, $+ ab$, being of the same kind as the multiplicand; that is, positive. Now, when the product, $+ ab$, is taken in connection with other quantities, the sign $+$ of the multiplier, a, shows that it is to be *added;* that is, written with its sign unchanged. $\therefore (+ b) \times (+ a) = + ab$.

2d. Let the factors be $- a$ and $- b$. Considering a as the multiplier, we are to take $- b$, a times, which gives $- ab$, a being considered as abstract in the operation, and the product, $- ab$, being of the same kind as the multiplicand; that is, negative. Now, when this product, $- ab$, is taken in connection with other quantities, the sign $-$ of the multiplier shows that it is to be subtracted; that is, written with its sign changed. $\therefore (- b) \times (- a) = + ab$.

3d. Let the factors be $- a$ and $+ b$. Considering a as the multiplier, we are to take $+ b$, a times, which gives $+ ab$, a being considered as abstract in the operation, and the product, $+ ab$, being of the same kind as the multiplicand; that is, positive. Now, when this product, $+ ab$, is taken in connection with other quantities, the sign $-$ of the multiplier shows that it is to be subtracted; that is, written with its sign changed. $\therefore (+ b) \times (- a) = - ab$.

4th. Let the factors be $+ a$ and $- b$. Considering a as the multiplier, we are to take $- b$, a times, which gives $- ab$, a being considered as abstract in the operation, and the product, $- ab$, being of the same kind as the multiplicand; that is, negative. Now, when this product, $- ab$, is taken in connection with other quantities, the sign $+$ of the multiplier shows that it is to be added; that is, written with its own sign. $\therefore (- b) \times (+ a) = - ab$. Q. E. D.

79. COR. 1.—*The product of any number of positive factors is positive.*

80. COR. 2.—*The product of an even number of negative factors is positive.*

81. COR. 3.—*The product of an odd number of negative factors is negative.*

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82. Prop. 3.—*The product of two or more factors consisting of the same quantity affected with exponents, is the common quantity with an exponent equal to the sum of the exponents of the factors.* That is $a^m \times a^n = a^{m+n}$; or $a^m \cdot a^n \cdot a^s = a^{m+n+s}$, etc., whether the exponents are integral or fractional, positive or negative.

DEM.—1st. *When the exponents are positive integers.* Let it be required to multiply a^m by a^n and a^s. $a^m = aaaa$ - - - - to m factors, $a^n = aaaaa$ - - - - to n factors, and $a^s = aaaaa$ - - - - to s factors. Hence the product, being composed of all the factors in the quantities to be multiplied together, contains $m + n + s$ factors each a, and hence is expressed a^{m+n+s}. Since it is evident that this reasoning can be extended to any number of factors, as $a^m \times a^n \times a^s \times a^r$, etc., etc., the proposition in this case is proved.

2d. *When the exponents are positive fractions.* Let it be required to multiply $a^{\frac{m}{n}}$ by $a^{\frac{c}{b}}$. Now $a^{\frac{m}{n}}$ means m of the n equal factors into which a is conceived to be resolved. If each of these n factors be resolved into b factors, a will be resolved into bn factors. Then, since $a^{\frac{m}{n}}$ contains m of the n equal factors of a, and each of these is resolved into b factors, m factors will contain bm of the bn equal factors of a. Hence $a^{\frac{m}{n}} = a^{\frac{bm}{bn}}$. In like manner $a^{\frac{c}{b}}$ may be shown equal to $a^{\frac{cn}{bn}}$; and $a^{\frac{m}{n}} \times a^{\frac{c}{b}} = a^{\frac{bm}{bn}} \times a^{\frac{cn}{bn}}$. This now signifies that a is to be resolved into bn factors, and $bm + cn$ of them taken to form the product. $\therefore a^{\frac{m}{n}} \times a^{\frac{c}{b}} = a^{\frac{bm}{bn}} \times a^{\frac{cn}{bn}} = a^{\frac{bm+cn}{bn}}$, or $a^{\frac{m}{n}+\frac{c}{b}}$, which proves the proposition for positive fractional exponents, since the same reasoning can be extended to any number of factors, as $a^{\frac{m}{n}} \times a^{\frac{c}{b}} \times a^{\frac{e}{d}}$, etc.

3d. *When the exponents are negative.* Let it be required to multiply a^{-m} by a^{-n}, m and n being either integral or fractional. By definition $a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n}$. Now, as fractions are multiplied by multiplying numerators together and denominators together, we have $\frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}}$ by part 1st of the demonstration. But this is the same as $a^{-(m+n)}$ or a^{-m-n}. $\therefore a^{-m} \times a^{-n} = a^{-m-n}$

EXAMPLES.

1. Prove as above that $81^{\frac{3}{2}} \times 81^{\frac{3}{4}} = 81^{\frac{18}{8}}$ and that $81^{\frac{18}{8}} = 81^{\frac{9}{4}}$.
2. Prove that $m^a \times m^b = m^{a+b}$.
3. Prove that $16^{-\frac{3}{4}} \times 16^{-\frac{1}{2}} = 16^{-\frac{5}{4}}$.
4. Prove that $25^{-\frac{1}{2}} \times 25^{\frac{1}{2}}$ is 1.
5. Prove that $a^{-2} \times a^3$ is a.

SCH.—The student must be careful to notice the difference between the signification of a fraction *used as an exponent*, and its common signification. Thus $\frac{2}{3}$ *used as an exponent* signifies that a number is resolved into 3 equal *factors*, and the *product* of 2 of them taken; whereas $\frac{2}{3}$ *used as a common fraction* signifies that a quantity is to be separated into 3 equal *parts*, and the *sum* of two of them taken.

83. Prob.—*To multiply monomials.*

RULE.—MULTIPLY THE NUMERICAL COEFFICIENTS AS IN THE DECIMAL NOTATION, AND TO THIS PRODUCT AFFIX THE LETTERS OF ALL THE FACTORS, AFFECTING EACH WITH AN EXPONENT EQUAL TO THE SUM OF ALL THE EXPONENTS OF THAT LETTER IN ALL THE

FACTORS. THE SIGN OF THE PRODUCT WILL BE + EXCEPT WHEN THERE IS AN ODD NUMBER OF NEGATIVE FACTORS; IN WHICH CASE IT WILL BE −.

DEM.—This rule is but an application of the preceding principles. Since the product is composed of all the factors of the given factors, and the order of arrangement of the factors in the product does not affect its value, we can write the product, putting the continued product of the numerical factors first, and then grouping the literal factors, so that like letters shall come together. Finally, performing the operations indicated, by multiplying the numerical factors as in the decimal notation, and the like literal factors by adding the exponents, the product is completed.

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84. Prob.*—To multiply two factors together when one or both are polynomials.*

RULE.—MULTIPLY EACH TERM OF THE MULTIPLICAND BY EACH TERM OF THE MULTIPLIER, AND ADD THE PRODUCTS.

DEM.—Thus, if any quantity is to be multiplied by $a + b - c$, if we take it a times (*i. e.* multiply by a), then b times, and add the results, we have taken it $a + b$ times. But this is taking it c too many times, as the multiplier required it to be taken $a + b$ *minus* c times. Hence we must multiply by c, and subtract this product from the sum of the other two. Now to subtract this product is simply to add it with its signs changed (***71***). But, regarding the − sign of c as we multiply, will change the signs of the product, and we can add the partial products as they stand, even without first adding the products by a and b. Q. E. D.

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85. THEO.—*The square of the sum of two quantities is equal to the square of the first, plus twice the product of the two, plus the square of the second.*

86. THEO.—*The square of the difference of two quantities is equal to the square of the first, minus twice the product of the two, plus the square of the second.*

87. THEO.—*The product of the sum and difference of two quantities is equal to the difference of their squares.*

The demonstration of these three theorems consists in multiplying $x + y$ by $x + y$, $x - y$ by $x - y$, and $x + y$ by $x - y$.

EXAMPLES.

1. Multiply together $3ax$, $-3a^2x^2$, $4by$, $-y^3$, and $2x^2y^2$.

2. Multiply together $3x^2$, $-mx^n$, $2m^s$, x^{-r}, -2, and $2x^m$.

3. Multiply together $40x^{\frac{1}{2}}$, $x^{\frac{2}{3}}y$ and $\frac{5}{2}x^{\frac{2}{3}}y^{\frac{1}{2}}$; also $3a^{\frac{1}{2}}b^{\frac{2}{3}}$, and $-3a^{\frac{2}{3}}b^{\frac{1}{2}}$.

4. Multiply $m^{\frac{2}{3}}$ by $m^{-\frac{1}{2}}$, a^{-n} by a^n, a^3b^{-x} by a^3b^x, $m^{-\frac{1}{n}}$ by $m^{\frac{1}{m}}$, $\sqrt[n]{a}$ by $\sqrt[m]{a}$, $\sqrt[4]{c^3}$ by $\sqrt[3]{c^4}$.

5. Multiply $3a - 2b$ by $a + 4b$.

6. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.

7. Multiply $m^4 + n^4 + o^4 - m^2n^2 - m^2o^2 - n^2o^2$ by $m^2 + n^2 + o^2$.

8. Multiply $a^m - a^n + a^2$ by $a^m - a$.

9. Multiply together $z - a$, $z - b$, $z - c$, $z - d$.

10. Multiply together $x + y$, $x - y$, $x^2 + xy + y^2$ and $x^2 - xy + y^2$.

SUG.—Try the factors in different orders, and compare the labor required.

11. Multiply $a^{\frac{m}{n}}b^{-\frac{t}{s}} - a^{\frac{m}{2n}}b^{-\frac{t}{2s}} + 1$ by $a^{\frac{m}{2n}}b^{-\frac{t}{2s}} + 1$.

12. Multiply $2a^{f-g}b^{1-n} + 3a^{n-1}b^m$ by $10a^{g-f+1}b^{n+1} - 5a^{p-q}b^{-m}$.

13. Square the following by the theorems (***85, 86***):

$1 + a$, $x - 2$, $3f + 3g$, $a^{-\frac{1}{2}} - a^{-\frac{3}{2}}b^2$, $x^n + x$, $\frac{a}{b} \pm \frac{b}{a}$, $x^{-x} + y^{-y}$, $\frac{2}{3}a^{\frac{2}{3}} - \frac{1}{3}a^{-\frac{1}{3}}$, $bx^{-1}y^{-\frac{1}{n}} - ay^{-1}x^{\frac{1}{n}}$, $2a^2b^{-(3-p)} + \frac{1}{2}xy^{-s}$.

14. Write the following products by (***87***):

$(3m^2 + 5n^2) \times (3m^2 - 5n^2)$, $(\sqrt{2}y^{\frac{1}{3}} + \sqrt[4]{3}z^{\frac{1}{2}}) \times (\sqrt{2}y^{\frac{1}{3}} - \sqrt[4]{3}z^{\frac{1}{2}})$, $(1 + \frac{2}{3}a) \times (1 - \frac{2}{3}a)$, $(99ax + 9\sqrt{ax}) \times (99ax - 9a^{\frac{1}{2}}x^{\frac{1}{2}})$.

15. Expand $(a + b + c)\ (a + b - c)\ (a - b + c)\ (-a + b + c)$.

Multiplication by Detached Coefficients.

88. In cases in which the terms of both multiplicand and multiplier contain the same letters, and can be so arranged that the exponents of the same letters shall vary in the successive terms of each according to the same law, a similar law will hold good in the product, and the multiplication can be effected by using the coefficients alone, in the first instance, and then writing the literal factors in the product according to the observed law. A few examples will make this clear:

1. Multiply $2a^3 - 3a^2x + 5ax^2 - x^3$ by $2a^2 - ax + 7x^2$.

OPERATION.

$$\begin{array}{r} 2-3+\ 5-\ 1 \\ 2-\ 1+\ 7 \\ \hline 4-6+10-\ 2 \\ -2+\ 3-\ 5+\ 1 \\ +14-21+35-7 \\ \hline 4-8+27-28+36-7 \end{array}$$

Prod., $4a^5 - 8a^4x + 27a^3x^2 - 28a^2x^3 + 36ax^4 - 7x^5$

2. Multiply $x^3 + 2x - 4$ by $x^2 - 1$.

Sug.—By writing these polynomials thus, $x^3 + 0x^2 + 2x - 4$, $x^2 + 0x - 1$, the law of the exponents in each case becomes evident. Hence we have,

$$\begin{array}{r} 1+0+2-4 \\ 1+0-1 \\ \hline 1+0+2-4 \\ -1-0-2+4 \\ \hline 1+0+1-4-2+4 \end{array}$$

Prod., $x^5 + 0x^4 + x^3 - 4x^2 - 2x + 4$, or $x^5 + x^3 - 4x^2 - 2x + 4$.

3. Multiply $3a^2 + 4ax - 5x^2$ by $2a^2 - 6ax + 4x^2$.

4. Multiply $2a^3 - 3ab^2 + 5b^3$ by $2a^2 - 5b^2$.

Sug.—The detached coefficients are $2 + 0 - 3 + 5$, and $2 + 0 - 5$.

5. Multiply $a^3 + a^2x + ax^2 + x^3$ by $a - x$.

6. Multiply $x^3 - 3x^2 + 3x - 1$ by $x^2 - 2x + 1$.

SECTION V.

DIVISION.

89. *Division* is the process of finding how many times one quantity is contained in another.

90. The problem of division may be stated: *Given the product of two factors and one of the factors, to find the other; and the sufficient reason for any quotient is, that multiplied by the divisor it gives the dividend.*

91. COR. 1.—*Dividend and divisor may both be multiplied or both be divided by the same number without affecting the quotient.*

92. COR. 2.—*If the dividend be multiplied or divided by any number, while the divisor remains the same, the quotient is multiplied or divided by the same.*

93. COR. 3.—*If the divisor be multiplied by any number while the dividend remains the same, the quotient is divided by that number; but if the divisor be divided, the quotient is multiplied.*

94. COR. 4.—*The sum of the quotients of two or more quantities divided by a common divisor, is the same as the quotient of the sum of the quantities divided by the same divisor.*

95. COR. 5.—*The difference of the quotients of two quantities divided by a common divisor, is the same as the quotient of the difference divided by the same divisor.*

These corollaries are direct consequences of the definition, and need no demonstration; but they should be amply illustrated.

96. DEF.—***Cancellation*** is the striking out of a factor common to both dividend and divisor, and does not affect the quotient, as appears from (***91***).

---

97. LEMMA 1.—*When the dividend is positive, the quotient has the same sign as the divisor; but when the dividend is negative, the quotient has an opposite sign to the divisor.*

98. LEMMA 2.—*When the dividend and divisor consist of the same quantity affected by exponents, the quotient is the common quantity with an exponent equal to the exponent in the dividend, minus that in the divisor.*

These lemmas are immediate consequences of the law of the signs and exponents in multiplication.

99. Cor. 1.—*Any quantity with an exponent* 0 *is* 1, *since it may be considered as arising from dividing a quantity by itself.*

Thus, x representing any quantity, and m any exponent, $x^m \div x^m = x^0 = 1$.

100. Cor. 2.—*Negative exponents arise from division when there are more factors of any number in the divisor than in the dividend.*

101. Cor. 3.—*A factor may be transferred from dividend to divisor (or from numerator to denominator of a fraction, which is the same thing), and vice versa, by changing the sign of its exponent.*

---

102. Prob. 1.—*To divide one monomial by another.*

RULE.—DIVIDE THE NUMERICAL COEFFICIENT OF THE DIVIDEND BY THAT OF THE DIVISOR AND TO THE QUOTIENT ANNEX THE LITERAL FACTORS, AFFECTING EACH WITH AN EXPONENT EQUAL TO ITS EXPONENT IN THE DIVIDEND MINUS THAT IN THE DIVISOR, AND SUPPRESSING ALL FACTORS WHOSE EXPONENTS ARE 0. THE SIGN OF THE QUOTIENT WILL BE + WHEN DIVIDEND AND DIVISOR HAVE LIKE SIGNS, AND − WHEN THEY HAVE UNLIKE SIGNS.

Dem.—The dividend being the product of divisor and quotient, contains all the factors of both; hence the quotient consists of all the factors which are found in the dividend and not in the divisor.

---

103. Prob. 2.—*To divide a polynomial by a monomial.*

RULE.—DIVIDE EACH TERM OF THE POLYNOMIAL DIVIDEND BY THE MONOMIAL DIVISOR, AND WRITE THE RESULTS IN CONNECTION WITH THEIR OWN SIGNS.

Dem.—This rule is simply an application of the corollaries (**94, 95**).

---

104. Def.—A polynomial is said to be arranged with reference to a certain letter when the term containing the highest exponent of that letter is placed first at the left or right, the term containing the next highest exponent next, etc., etc.

105. Prob. 3.—*To perform division when both dividend and divisor are polynomials.*

RULE.—HAVING ARRANGED DIVIDEND AND DIVISOR WITH REFERENCE TO THE SAME LETTER, DIVIDE THE FIRST TERM OF THE DIVIDEND BY THE FIRST TERM OF THE DIVISOR FOR THE FIRST TERM OF THE QUOTIENT. THEN SUBTRACT FROM THE DIVIDEND THE PRODUCT OF THE DIVISOR INTO THIS TERM OF THE QUOTIENT, AND BRING DOWN AS MANY TERMS TO THE REMAINDER AS MAY BE NECESSARY TO FORM A NEW DIVIDEND. DIVIDE AS BEFORE, AND CONTINUE THE PROCESS TILL THE WORK IS COMPLETE.

DEM.—The arrangement of dividend and divisor according to the same letter enables us to find the term in the quotient containing the highest (or lowest if we put the lowest power of the letter first in our arrangement) power of the same letter, and so on for each succeeding term.

The other steps of the process are founded on the principle, that the product of the divisor into the several parts of the quotient is equal to the dividend. Now by the operation, the product of the divisor into the *first* term of the quotient is subtracted from the dividend; then the product of the divisor into the *second* term of the quotient; and so on, till the product of the divisor into each term of the quotient, that is, the product of the divisor into the *whole* quotient, is taken from the dividend. If there is no remainder, it is evident that this product is *equal* to the dividend. If there *is* a remainder, the product of the divisor and quotient is equal to the whole of the dividend *except* the remainder. And this remainder is not included in the parts subtracted from the dividend, by operating according to the rule.

SCH.—This process of division is strictly analogous to "Long Division" in common arithmetic. The arrangement of the terms corresponds to the regular order of succession of the thousands, hundreds, tens, units, etc., while the other processes are precisely the same in both.

EXAMPLES.

1. Divide $m^{\frac{3}{4}}$ by $m^{\frac{1}{3}}$, $n^{\frac{m}{n}}$ by n^{-1}, $(ab)^{2m}$ by $(ab)^{\frac{1}{n}}$, a^3 by a^5, a^{-2} by a^3, $x^{-\frac{1}{3}}$ by x^{-2}, x^{-2} by $x^{-\frac{1}{3}}$.

2. Free $\frac{a^{-2}b^2}{x^{-2}y^3}$, $\frac{2a^{-\frac{1}{2}}x^{-\frac{2}{3}}y}{3m^2n^{-1}x^2}$, and $\frac{5cd^{-1}bx^{-4}}{8a^{-2}xy^{-1}z}$ from negative exponents, and explain the process.

3. Divide $15ay^2$ by $3ay$, $8a^4b^3c^2d$ by $4a^2b^2c^2$, $3a^{\frac{1}{3}}b^{\frac{2}{3}}$ by $a^{\frac{1}{3}}b^{\frac{2}{3}}$, $-35a^4bx^{\frac{1}{2}}$ by $7a^2bx$, $-20a^{\frac{3}{2}}b^{\frac{7}{2}}c$ by $-40ab^4c$, y^a by y^b, $-y^a$ by y^{-b}, $12a^sb^{p-q}y$ by $-7a^{-s}b^{q-p}y^{-n}$, $-4a^{\frac{2}{3}}b^{-1}c^2$ by $-12a^{-\frac{2}{3}}bc^{2-n}$, $a^{p-q+1}b^{r-s}c^{\frac{3}{2}}$ by $a^{q-p+1}b^{r+s}c^2$, and $x^{\frac{m}{n}}y^{-1}$ by $x^{\frac{n}{m}}y^{-1}$.

4. Divide $9a^3k^2 - 12a^2k^3 + 3a^2k^2$ by $3ak$, $11x^4y^4a^2b + 121x^3y^3 - 44x^6y^nab^2$ by $11x^3y^3$, $15ax^3 - 15a^3x + 3ax$ by $-5ax$, $4a^{10}m^4 - 12a^{-10}m^8 + 5280$ by $-12a^{-10}$, $209x^{\frac{2}{3}}y^m - 247xy^{m+1}$ by $19xy$, $y^{\frac{5}{8}} + 3a^2y^{\frac{3}{8}} - 2y^{\frac{1}{8}}$ by $y^{\frac{1}{8}}$, $b^{1+n} - b^{1+2n} - b^{1+3n} - b^{1+4n}$ by b^{3n}, $ax^{\frac{m}{n}} - 2ax^{-\frac{n}{m}-1} + 3ax$ by ax^{s+1}.

5. Divide $4x^2 - 28xy + 49y^2$ by $2x - 7y$.

6. Divide $6x^4 - 13ax^3 + 13a^2x^2 - 13a^3x - 5a^4$ by $2x^2 - 3ax - a^2$.

7. Divide $x^3 + y^3 + 3xy - 1$ by $x + y - 1$.

8. Divide $a^6b^{12} - 64$ by $ab^2 - 2$, $x - 4a^{\frac{3}{2}}$ by $x^{\frac{1}{2}} - 2a^{\frac{3}{4}}$.

9. Divide $xy - a$ by $x^{\frac{1}{3}}y^{\frac{1}{3}} - a^{\frac{1}{3}}$, $243a^5 + 1024$ by $4 + 3a$.

10. Divide $y^5 - \frac{23}{30}y^4 + \frac{31}{10}y^3 - \frac{7}{3}y^2 - \frac{181}{18}y + \frac{5}{3}$ by $y^2 - \dfrac{y}{6} + 5$.

11. Divide $1 + 2x^2 - 7x^4 - 16x^6$ by $1 + 2x + 3x^2 + 4x^3$.

12. Divide $(x^2 - y^2)^3$ by $(x - y)^3$, $a^3 + b^{-3}$ by $a + b^{-1}$.

13. Divide $y^4 - \dfrac{1}{y^4}$ by $y - \dfrac{1}{y}$.

14. Divide 1 by $1 - x^2$, also by $1 + x^2$, $1 + x$, and by $1 - x$.

15. Divide $a^{1+n} + a^nb + ab^n + b^{1+n}$ by $a^n + b^n$.

16. Divide $a^{3m-2n}b^{2p}c - a^{2m+n-1}b^{1-p}c^n + a^{-n}b^{-1}c^m + a^{2m-n}b^{3p+2}c^{n} - a^{2m+2n-1}b^3c^{2n-1} + b^{p+1}c^{m+n-1}$ by $a^{-n}b^{-p-1} + bc^{n-1}$.

17. Divide $m^{m+1} + amn^{an} + nm^m + an^{an+1}$ by $m + n$.

18. Divide $mn(x^5 + 1) + (n^2 + m^2)(x^4 + x) + (n^2 + 2nm)(x^3 + x^2)$ by $nx^2 + mx + n$.

19. Divide $hkx^4 + 2(h - k)x^3 - (h^2 + 4 - k^2)x^2 + 2(h + k)x - hk$ by $kx^2 - h + 2x$.

20. Divide $x + y + z - 3\sqrt[3]{xyz}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}$.

Division by Detached Coefficients.

106. Division by detached coefficients can be effected in the same cases as multiplication (***88***). The student will be able to trace the process and see the reason from an example.

1. Divide $10a^4 - 27a^3x + 34a^2x^2 - 18ax^3 - 8x^4$ by $2a^2 - 3ax + 4x^2$.

OPERATION.

$$
\begin{array}{r|l}
2-3+4)\,10-27+34-18-8 & 5 \quad -6 \quad -2 \\
10-15+20 & \overline{5a^2-6ax-2x^2} \quad \textit{Quot.} \\
\hline
-12+14-18 & \\
-12+18-24 & \\
\hline
-4+6-8 & \\
-4+6-8 & \\
\hline
\end{array}
$$

2. Divide $x^4 - 3ax^3 - 8a^2x^2 + 18a^3x - 8a^4$ by $x^2 + 2ax - 2a^2$.
3. Divide $6a^4 - 96$ by $3a - 6$.

SUG.—The detached coefficients are $6 + 0 + 0 + 0 - 96$ and $3 - 6$.

4. Divide $3y^3 + 3xy^2 - 4x^2y - 4x^3$ by $x + y$.
5. Divide $x^7 + y^7$ by $x + y$; also $x^4 - y^4$ by $x^2 - y^2$.

SYNTHETIC DIVISION.

107. When division by detached coefficients is practicable, as in the examples in the last article, the operation may be very much condensed by an arrangement of terms first proposed by W. G. Horner, Esq., of Bath, Eng., which is hence called Horner's method of synthetic division. A careful inspection of the OPERATION under Ex. 1, in the last article, will acquaint the student with the process.

OPERATION.

$$
\begin{array}{r|r}
2 & 10-27+34-18-8 \\
+3 & +15-20+24+8 \\
-4 & -18-\ 6 \\
\hline
 & 5 \quad -6 \quad -2 \\
 & \overline{5a^2-6ax-2x^2}, \ \textit{Quot.}
\end{array}
$$

EXPLANATION OF OPERATION.—Arrange the coefficients of the divisor in a vertical column at the left of the dividend, changing the signs of all after the first. Draw a line underneath the whole under which to write the coefficients of the quotient.

The first coefficient of the quotient is found evidently by dividing the first of the dividend by the first of the divisor, and in this case is 5. As the first term of the dividend is always destroyed by this operation, we need give it (10) no farther consideration. Now, multiplying the other coefficients after the first (*i. e.* + 3 and − 4) *with their signs changed,* by 5, we have + 15 and − 20, which are to be *added* (?) to − 27 and + 34. Hence we write the former under the latter. The first term* of the second partial dividend can be formed mentally by *adding* (?) + 15 to − 27, and the next term of the quotient by dividing this sum (− 12) by 2. Hence − 6 is the second term of

* Strictly, the "coefficient of;" but this form is used for brevity.

the quotient. (We did not *add* (?) -20 to $+34$, because there is more to be taken in before the first term of the next partial dividend is formed.)

Having found the second term of the quotient (-6), we multiply the terms of the divisor, except the first, (with their signs changed) by -6, and write the results, -18 and $+24$, under the third and fourth of the dividend, to which they are to be *added* (?). Now we have all that is to be *added** to $+34$ (viz., -20 and -18) in order to obtain the first term of the next partial dividend. Hence, adding, we get -4. which divided by 2 gives -2 as the next term of the quotient. Multiplying all the terms of the divisor except the first, as before, we have -6 and $+8$, which fall under -18 and -8. Now adding $+24$ and -6 to -18, nothing remains. So also $+8-8=0$, and the work is complete, as far as the coefficients of the quotient are concerned.

2. Divide $x^6 - 5x^5 + 15x^4 - 24x^3 + 27x^2 - 13x + 5$ by $x^4 - 2x^3 + 4x^2 - 2x + 1$.

OPERATION.

1	1	−5	+15	−24	+27	−13	+5
+2		+2	−4	+2	−1	+3	−5
−4			−6	+12	−6	+10	
+2				+10	−20		
−1							
	1	−3	+5	0	0	0	0

Quot., $x^2 - 3x + 5$

3. Divide $4y^6 - 24y^5 + 60y^4 - 80y^3 + 60y^2 - 24y + 4$ by $2y^2 - 4y + 2$.

4. Divide $x^7 - y^7$ by $x - y$; also 1 by $1 - x$.

5. Will $x + 2$ divide $x^4 + 2x^3 - 7x^2 - 20x + 12$ without a remainder? Will $x-3$?

6. Will $x + 3$, or $x - 3$, divide $x^4 - 6x^2 - 16x + 21$ without a remainder? Will $x + 7$, or $x - 7$?

* The student will not fail to see that this addition is equivalent to the ordinary subtraction since the signs of the terms have been changed.

CHAPTER II.

FACTORING.

SECTION I.

FUNDAMENTAL PROPOSITIONS.

108. The Factors of a number are those numbers which multiplied together produce it. A *Factor* is, therefore, a *Divisor.* A *Factor* is also frequently called a *measure,* a term arising in Geometry.

109. A Common Divisor is a common integral factor of two or more numbers. The *Greatest Common Divisor* of two or more numbers is the greatest common integral factor, or the product of all the common integral factors. *Common Measure* and *Common Divisor* are equivalent terms.

110. A Common Multiple of two or more numbers is an integral number which contains each of them as a factor, or which is divisible by each of them. The *Least Common Multiple* of two or more numbers is the least integral number which is divisible by each of them.

111. A Composite Number is one which is composed of integral factors different from itself and unity.

112. A Prime Number is one which has no integral factor other than itself and unity.

113. Numbers are said to be *Prime to each other* when they have no common integral factor other than unity.

Sch. 1.—The above definitions and distinctions have come into use from considering Decimal Numbers. They are applicable to literal numbers only in an accommodated sense. Thus, in the general view which the literal notation requires, all numbers are composite in the sense that they can be factored; but as to whether the factors are greater or less than unity, integral or fractional, we cannot affirm.

---

114. Prop. 1.—*A monomial may be resolved into literal factors by separating its letters into any number of groups, so that the sum of all the exponents of each letter shall make the exponent of that letter in the given monomial.*

---

115. Prop. 2.—*Any factor which occurs in every term of a polynomial can be removed by dividing each term of the polynomial by it.*

---

116. Prop. 3.—*If two terms of a trinomial are* POSITIVE *and the third term is twice the product of the square roots of these two, and* POSITIVE, *the trinomial is the square of the* SUM *of these square roots. If the third term is* NEGATIVE, *the trinomial is the square of the* DIFFERENCE *of the two roots.*

---

117. Prop. 4.—*The difference between two quantities is equal to the product of the sum and difference of their square roots.*

---

118. Prop. 5.—*When one of the factors of a quantity is given, to find the other, divide the given quantity by the given factor, and the quotient will be the other.*

---

119. Prop. 6.—*The difference between any two quantities is a divisor of the* DIFFERENCE *between the same powers of the quantities.*

The SUM *of two quantities is a divisor of the* DIFFERENCE *of the same* EVEN *powers, and the* SUM *of the same* ODD *powers of the quantities.*

DEM.—Let x and y be any two quantities and n any positive integer. *First,* $x - y$ divides $x^n - y^n$. *Second,* if n is *even,* $x + y$ divides $x^n - y^n$. *Third,* if n is *odd,* $x + y$ divides $x^n + y^n$.

FIRST.

Taking the first case, we proceed in form with the division, till four of the terms of the quotient (enough to determine the law) are found. We find that each remainder consists of two terms, the second of which, $-y^n$, is the second term of the dividend constantly brought down unchanged; and the first contains x with an exponent decreasing by unity in each successive remainder, and y with an exponent *increasing* at the same rate that the exponent of x *decreases*. At this rate the exponent of x in the nth remainder becomes 0, and that of y, n. Hence the nth remainder is $y^n - y^n$ or 0; and the division is exact.

$$
\begin{array}{l}
x - y)x^n - y^n \quad (x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + \text{etc.} \\
\quad x^n - x^{n-1}y \\
\quad\quad x^{n-1}y - y^n \\
\quad\quad x^{n-1}y - x^{n-2}y^2 \\
\quad\quad\quad x^{n-2}y^2 - y^n \\
\quad\quad\quad x^{n-2}y^2 - x^{n-3}y^3 \\
\quad\quad\quad\quad x^{n-3}y^3 - y^n \\
\quad\quad\quad\quad x^{n-3}y^3 - x^{n-4}y^4 \\
\quad\quad\quad\quad\quad x^{n-4}y^4 - y^n
\end{array}
$$

SECOND AND THIRD.

$$
\begin{array}{l}
x + y)x^n \pm y^n \quad (x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3, \text{ etc.} \\
\quad x^n + x^{n-1}y \\
\quad\quad - x^{n-1}y \pm y^n \\
\quad\quad - x^{n-1}y - x^{n-2}y^2 \\
\quad\quad\quad x^{n-2}y^2 \pm y^n \\
\quad\quad\quad x^{n-2}y^2 + x^{n-3}y^3 \\
\quad\quad\quad\quad - x^{n-3}y^3 \pm y^n \\
\quad\quad\quad\quad - x^{n-3}y^3 - x^{n-4}y^4 \\
\quad\quad\quad\quad\quad x^{n-4}y^4 \pm y^n
\end{array}
$$

Taking $x + y$ for a divisor, we observe that the exponent of x in the successive remainders decreases, and that of y increases the same as before. But now we observe that the first term of the remainder is $-$ in the *odd* remainders, as the 1st, 3d, 5th, etc., and $+$ in the *even* ones, as the 2d, 4th, 6th, etc. Hence if n is *even, and the second term of the dividend is* $-y^n$, the nth remainder is $y^n - y^n$ or 0, and the division is exact. Again, if n is *odd*, and the second term of the dividend is $+y^n$, the nth remainder is $-y^n + y^n$, or 0, and the division is exact. Q. E. D.

120. COR.—*The last proposition applies equally to cases involving fractional or negative exponents.*

DEM.—Thus, $x^{\frac{1}{5}} - y^{\frac{1}{5}}$ divides $x^{\frac{4}{5}} - y^{\frac{4}{5}}$, since the latter is the difference between the 4th powers of $x^{\frac{1}{5}}$ and $y^{\frac{1}{5}}$. So in general $x^{-\frac{n}{m}} - y^{-\frac{s}{r}}$ divides $x^{-\frac{an}{m}} - y^{-\frac{as}{r}}$, a being any positive integer. This becomes evident by putting $x^{-\frac{n}{m}} = v$, and $y^{-\frac{s}{r}} = w$; whence $x^{-\frac{an}{m}} = v^a$, and $y^{-\frac{as}{r}} = w^a$. But $v^a - w^a$ is divisible by $v - w$, hence $x^{-\frac{an}{m}} - y^{-\frac{as}{r}}$ is divisible by $x^{-\frac{n}{m}} - y^{-\frac{s}{r}}$.

121. Prop. 7.—*A trinomial can be resolved into two binomial factors, when one of its terms is the product of the square root of one of the other two, into the sum of the factors of the remaining term. The two factors are respectively the algebraic sum of this square root, and each of the factors of the third term.*

ILL.—Thus, in $x^2 + 7x + 10$, we notice that $7x$ is the product of the square root of x^2, and $2 + 5$ (the sum of the factors of 10). The factors of $x^2 + 7x + 10$ are $x + 2$ and $x + 5$. Again, $x^2 - 3x - 10$, has for its factors $x + 2$ and $x - 5$, $-3x$ being the product of the square root of x^2 (or x), and the sum of -5 and 2, (or -3), which are factors of -10. Still again, $x^2 + 3x - 10 = (x - 2)(x + 5)$, determined in the same manner.

DEM.—The truth of this proposition appears from considering the product of $x + a$ by $x + b$, which is $x^2 + (a + b)x + ab$. In this product, considered as a trinomial, we notice that the term $(a + b)x$ is the product of $\sqrt{x^2}$ and $a + b$, the sum of the factors of ab. In like manner $(x + a)(x - b) = x^2 + (a - b)x - ab$, and $(x - a)(x - b) = x^2 - (a + b)x + ab$, both of which results correspond to the enunciation. Q. E. D.

[NOTE.—In application, this proposition requires the solution of the problem: Given the sum and product of two numbers to find the numbers, the complete solution of which cannot be given at this stage of the pupil's progress. It will be best for him to rely, at present, simply upon inspection.]

---

122. Prop. 8.—*We can often detect a factor by separating a polynomial into parts.*

Ex. Factor $x^2 + 12x - 28$.

SOLUTION.—The form of this polynomial suggests that there may be a binomial factor in it, *or in a part of it*. Now $x^2 - 4x + 4$ is the square of $x - 2$, and $(x^2 - 4x + 4) + (16x - 32)$ makes $x^2 + 12x - 28$. But $(x^2 - 4x + 4) + (16x - 32) = (x - 2)(x - 2) + (x - 2)16 = (x - 2)(x - 2 + 16) = (x - 2)(x + 14)$. Whence $x - 2$, and $x + 14$ are seen to be the factors of $x^2 + 12x - 28$.

MISCELLANEOUS EXAMPLES.

1. Factor $7fg^2y - 28f^2gy^2 + 42f^3gy$, $4x^2y^3 - 7x^2y^4 + 12xy^5$.
2. Factor $m^4 - n^4$, $1 - 2\sqrt{x} + x$, $256a^4 + 544a^2 + 289$, $1 - c^3$.
3. Factor $x^2 - x - 72$, $y^6 - z^4$, $a^3 + b^3$, $\frac{a^2}{b^2} + \frac{b^2}{a^2} - 2$, $a^2 + 23a + 22$.
4. Factor $\frac{25}{m^2} - \frac{40}{mx^2} + \frac{16}{x^4}$, $c^6 - d^6$, $c^{-6} - d^{-6}$, $c^6 - d^{-6}$.

5. Factor $a^{\frac{2}{3}} - m^{\frac{2}{3}}$, $4x^4 - 5y^{-4}$, $\frac{a^4}{x^4} - b^{10}$, $x^2 + 22x - 7623$.

6. Factor $x^n - 1$, $507m^4 + 1326m^2n^{\frac{3}{2}} + 867n^3$, $\sqrt{a} - \sqrt{b}$.

7. Factor $x^2 - 2ax - a^2$, $a^m \pm 4b^4\sqrt{a^m c^m} + 4b^8c^m$, $x^2 + \sqrt{x} + 2x^{\frac{5}{4}}$.

8. Factor $\frac{9}{49}a^{4m} - \frac{2}{21}a^{2m}b^{2n+2} + \frac{1}{81}b^{4n+4}$, $3a + 3b - 6\sqrt{ab}$.

9. Resolve x into two equal factors; also two unequal factors.

10. Resolve $38x^4y^2z^{\frac{1}{2}} - 3\sqrt{y^4z}$ into two factors of which one is $2y^2\sqrt{z}$.

11. Resolve $121a^{\frac{4}{7}}b^{\frac{3}{7}}c^{\frac{2}{5}}$ into two equal factors; also into four equal factors.

12. Remove the factor $7(ak^3)^{\frac{1}{3}}$ from $84a^{\frac{2}{3}}k^4$.

13. Remove the factor $\frac{m^4}{n^4} + \frac{7c^{-2}}{3d^{-\frac{3}{2}}}$ from $m^8n^{-8} - \frac{49d^3}{9c^4}$.

14. Remove the factor $a^4 - a^3b + a^2b^2 - ab^3 + b^4$ from $a^5 + b^5$.

15. Factor $15a + 5ax - x - 3$, $21abcd - 28cdxy + 15abmn - 20mnxy$, $21x^2 + 23xy - 20y^2$, $12a^2x^{\frac{3}{2}} - 12a^2x^{\frac{3}{4}} + 3a^2$.

16. Factor $3x^3 - 12x^3y^2 - 4y^2 + 1$, $72cd^2m^3 - 84cd^3m^2 + 96c^2d^2m^2$.

17. The terms of a trinomial are $30ab$, $9a^2$ and $25b^2$. What sign must be given to each that the trinomial may be factored?

18. The terms of a trinomial are $-9a$, $12\sqrt{a}$ and 4. What must be the signs of the last two terms that the trinomial may be factored?

19. Is $a^{\frac{2n}{5}} - b^{2n}$ exactly divisible by $a^{\frac{1}{5}} - b$ or $a^{\frac{1}{5}} + b$?

20. Is $m^5 - n^5$ exactly divisible by $\sqrt{m} - \sqrt{n}$? by $\sqrt{m} + \sqrt{n}$? by $\sqrt[3]{m} \pm \sqrt[3]{n}$?

21. Is $x^{101} + y^{101}$ exactly divisible by $x + y$? by $x - y$?

22. Is $x^{2079} + y^{2079}$ exactly divisible by $x^7 - y^7$? by $x^7 + y^7$?

23. What is the quotient of $(ky^{\frac{7}{3}} + mz^{\frac{7}{5}}) \div (k^{\frac{1}{7}}\sqrt[3]{y} + \sqrt[7]{m}\,z^{\frac{1}{5}})$?

24. What is the quotient of $(x^{\frac{1}{2}} + y^{\frac{1}{2}}) \div (x^{\frac{1}{14}} + y^{\frac{1}{14}})$?

25. Write the following quotients: $(a^8 + b^8) \div (a^2 + b^2)$; $(x^{2m} - z^{2m}) \div (x - z)$; $(x^{2m} - z^{2m}) \div (x + z)$; $(x^{2m+1} + z^{2m+1}) \div (x + z)$, m being a positive integer.

26. Factor $x^2 + ax + x + a$, $1 - a$, $1 + a$, $\frac{1}{x^{10}} - \frac{100}{y^{100}}$ and $x^2 - x - 9900$.

27. Factor $10a\left[\frac{x^4}{y^2} + \frac{y^2}{x^4}\right] - 20a$, $4x + 4x^{\frac{1}{2}} + 1$ and $36a^m - 5b^{2n}$.

28. $x^3 - x^2 - 2x + 2$, $6x^3 - 7ax^2 - 20a^2x$, $x^{2m} + 31x^m - 32$.

SECTION II.

GREATEST OR HIGHEST COMMON DIVISOR.

123. DEF.—It is scarcely proper to apply the term Greatest Common Divisor to literal quantities, for the values of the letters not being fixed, or specific, *great* or *small* cannot be affirmed of them. Thus, whether a^3 is greater than a, depends upon whether a is greater or less than 1, to say nothing of its character as positive or negative. So, also, we cannot with propriety call $a^3 - y^3$ *greater* than $a - y$. If $a = \frac{1}{2}$, and $y = \frac{1}{4}$, $a^3 - y^3 = \frac{7}{64}$, and $a - y = \frac{1}{4}$; $\therefore$ in this case $a^3 - y^3 < a - y$. Again, if a and y are both greater than 1, but $a < y$, $a^3 - y^3$ though *numerically* greater than $a - y$ is *absolutely* less, since *it is a greater negative.*

Instead of speaking of G. C. D. in case of literal quantities, we should speak of the *Highest Common Divisor,* since what is meant is the divisor which is of the highest degree with reference to the letter of arrangement.

[NOTE.—The general rule for finding the Greatest or Highest Common Divisor is founded upon the four following lemmas.]

124. LEMMA 1.—*The Greatest or Highest C. D. of two or more numbers is the product of their common prime factors.*

DEM.—Since a factor and a divisor are the same thing, all the common factors are all the common divisors. And, since the product of any number of factors of a number is a divisor of that number, the product of *all* the *common prime factors* of two or more numbers is a *common divisor* of those numbers. Moreover, this product is the *Greatest* or *Highest* C. D., since no other factor can be introduced into it without preventing its measuring (dividing), at least, one of the given numbers. Q. E. D.

Examples.

1. What is the G. C. D. of 72, 84, and 180?

Solution.—Resolve the numbers into their prime factors, and take the product of those which are common to all.

2. Find the G. C. D. of 48, 204, and 228.
3. Find the G. C. D. of 81, 123, and 315.
4. Find the Highest C. D. of $8x^3yz^2$ and $15x^2y$.

Solution.—Here we see that x, x, and y are all the literal factors common to both; and since 8 and 15 have no common factor, $x \times x \times y$ is the Highest C. D.

5. Find the H. C. D. of $14k^3l^8m^5$ and $30k^8l^5m^2n^2$.

6. Find the H. C. D. of $8a^2bc$, $18a^3b^2$, and $26a^2b^{\frac{3}{2}}mn$.

7. Find the H. C. D. of $7x^{\frac{2}{3}}y^{-\frac{2}{3}}z^n$ and $4xy^{-2}z^{n+1}$.

8. Find the H. C. D. of $5a^2x^2y - 10ax^3y + 5ax^4y$ and $3a^2x^2y - 3x^2y^2$.

9. Find the H. C. D. of $x^2 - x - 12$ and $x^3 - x^2 - 9x + 9$.

Solution.—$x^2 - x - 12 = (x - 4)(x + 3)$ (*121*). $x^3 - x^2 - 9x + 9 = x^2(x - 1) - 9(x - 1) = (x^2 - 9)(x - 1) = (x - 3)(x + 3)(x - 1)$. Now we see that $x + 3$ is a common divisor of the two polynomials, and since it is the *only* divisor common to both, it is the H. C. D.

10. Find the H. C. D. of $4b^2x^3 - 12b^2x^2 + 12b^2x - 4b^2$ and $4b^2x^3 - 8b^2x^2 - 4b^2x + 8b^2$.

125. Sch.—The difficulty of factoring renders this process impracticable in many cases. There is a more general method. But, in order to demonstrate the rule, we require three additional lemmas.

126. Lemma 2.—*A polynomial of the form* $Ax^n + Bx^{n-1} + Cx^{n-2}$ - - - - $Ex + F$, *which has no common factor in every term, has no divisor of its own degree except itself.*

Dem.—1st. Such a polynomial cannot have one factor of the nth degree—its own—with reference to the letter of arrangement, and another which contains the letter of arrangement, for the product of two *such* factors would be of a higher (or different) degree from the given polynomial.

2d. It cannot have a factor of the nth degree with reference to the letter of arrangement, and another factor which does not contain that letter, for this last factor would appear as a common factor in every term, which is contrary to the hypothesis. Q. E. D.

127. LEMMA 3.—*A divisor of any number is a divisor of any multiple of that number.*

ILL.—This is an axiom. If a goes into b, q times, it is evident that it goes into n times b, or nb, n times q, or nq times.

128. LEMMA 4.—*A common divisor of two numbers is a divisor of their sum and also of their difference.*

DEM.—Let a be a C. D. of m and n, going into m, p times, and into n, q times. Then $(m \pm n) \div a = p \pm q$. Q. E. D.

129. *Prob.*—*To find the H. C. D. of two polynomials without the necessity of resolving them into their prime factors.*

RULE.—1st. ARRANGING THE POLYNOMIALS WITH REFERENCE TO THE SAME LETTER, AND UNITING INTO SINGLE TERMS THE LIKE POWERS OF THAT LETTER, REMOVE ANY COMMON FACTOR OR FACTORS WHICH MAY APPEAR IN ALL THE TERMS OF BOTH POLYNOMIALS, RESERVING THEM AS FACTORS OF THE H. C. D.

2d. REJECT FROM EACH POLYNOMIAL ALL OTHER FACTORS WHICH APPEAR IN EACH TERM OF EITHER.

3d. TAKING THE POLYNOMIALS, THUS REDUCED, DIVIDE THE ONE WITH THE GREATEST EXPONENT OF THE LETTER OF ARRANGEMENT, BY THE OTHER, CONTINUING THE DIVISION TILL THE EXPONENT OF THE LETTER OF ARRANGEMENT IS LESS IN THE REMAINDER THAN IN THE DIVISOR.

4th. REJECT ANY FACTOR WHICH OCCURS IN EVERY TERM OF THIS REMAINDER, AND DIVIDE THE DIVISOR BY THE REMAINDER AS THUS REDUCED, TREATING THE REMAINDER AND LAST DIVISOR AS THE FORMER POLYNOMIALS WERE. CONTINUE THIS PROCESS OF REJECTING FACTORS FROM EACH TERM OF THE REMAINDER, AND DIVIDING THE LAST DIVISOR BY THE LAST REMAINDER TILL NOTHING REMAINS.

IF, AT ANY TIME, A FRACTION WOULD OCCUR IN THE QUOTIENT, MULTIPLY THE DIVIDEND BY ANY NUMBER WHICH WILL AVOID THE FRACTION.

THE LAST DIVISOR MULTIPLIED BY ALL THE FIRST RESERVED COMMON FACTORS OF THE GIVEN POLYNOMIALS, WILL BE THE H. C. D. SOUGHT.

DEM.—Let A and B represent any two polynomials whose H. C. D. is sought.

1st. Arranging A and B with reference to the same letter, for convenience in dividing, and also to render common factors more readily discernible, if any common factors appear, they can be removed and reserved as factors of the H. C. D., since the H. C. D. consists of all the common factors of A and B.

2d. Having removed these common factors, call the remaining factors C and D. We are now to ascertain what common factors there are in C and D, or to find their H. C. D. As this H. C. D. consists of only the common factors, we can *reject* from each of the polynomials, C and D, any factors which are not common. Having done this, call the remaining factors E and F.

3d. Suppose polynomial E to be of lower degree with respect to the letter of arrangement than F. (If E and F are of the same degree, it is immaterial which is made the divisor in the subsequent process.) Now, as E is its own only divisor of its own degree (LEM. 2), if it divides F, it is the H. C. D. of the two. If, in attempting to divide F by E to ascertain whether it is a divisor, fractions arise, F can be multiplied by any number not a factor in E (and E has no monomial factor), since the common factors of E and F would not be affected by the operation. Call such a multiple of F, if necessary, F′. Then the H. C. D. of E and F′, is the H. C. D. of E and F. If, now, E divides F′, it is the H. C. D. of E and F. Trying it, suppose it goes Q times, with a remainder, R.

4th. Any divisor of E and F′ is a divisor of R, since $F' - QE = R$, and any divisor of a number divides any multiple of that number (LEM. 3), and a divisor of two numbers divides their difference. The H. C. D. divides E, hence it divides QE, and, as it also divides F′, it divides the difference between F′ and QE, or R. Therefore the H. C. D. of E and F′, is also a divisor of E and R, and cannot be of higher degree than R.

5th. We now repeat the reasoning of the 3d and 4th paragraphs concerning E and F, with reference to E and R. Thus, R is by hypothesis of lower degree than E; hence, dividing E by it, rejecting any factor not common to both, or introducing any one into E, which may be necessary to avoid fractions, we ascertain whether R is a divisor of E. If it is, it divides F′, since $F' = R + QE$ (LEM. 3, 4), and hence is the H. C. D. of E and F′.

6th. Proceeding thus, till two numbers are found, one of which divides the other, the last divisor is the H. C. D. of E and F, since at every step we show that the H. C. D. is a divisor of the two numbers compared, and the last divisor is its own H. D.

7th. Finally, we have thus found all the common factors of A and B, the product of which is their H. C. D. Q. E. D.

EXAMPLES.

1. Find the H. C. D. of $12a^2b^2 + 3b^2y^2 - 15ab^2y + 12a^2by + 3by^3 - 15aby^2$, and $6a^3b^2 - 6a^2b^2y - 2b^2y^3 + 2ab^2y^2 + 6a^3by - 6a^2by^2 - 2by^4 + 2aby^3$.

OPERATION.

$12a^2b^2 + 3b^2y^2 - 15ab^2y + 12a^2by + 3by^3 - 15aby^2$ - - - - - - - - - - (A).

$6a^3b^2 - 6a^2b^2y - 2b^2y^3 + 2ab^2y^2 + 6a^3by - 6a^2by^2 - 2by^4 + 2aby^3$ - - (B).

$4a^2b + by^2 - 5aby + 4a^2y + y^3 - 5ay^2$ - - - - - - - - - - - - - - - (C).

$3a^3b - 3a^2by - by^3 + aby^2 + 3a^3y - 3a^2y^2 - y^4 + ay^3$ - - - - - - - - (D).

$(4b + 4y)a^2 - (5by + 5y^2)a + (by^2 + y^3)$ - - - - - - - - - - - - - - - (E).

$(3b + 3y)a^3 - (3by + 3y^2)a^2 + (by^2 + y^3)a - (by^3 + y^4)$ - - - - - - - - (F).

(G) (H)

$4a^2 - 5ya + y^2)\ 3a^3 - 3ya^2 + y^2a - y^3$

4

(I) - - - - $12a^3 - 12ya^2 + 4y^2a - 4y^3(3a$

(K) - - - - $12a^3 - 15ya^2 + 3y^2a$

(L) - - - - - - - $3ya^2 + y^2a - 4y^3$

4

(M) - - - - - - - - $12ya^2 + 4y^2a - 16y^3(3y$

(N) - - - - - - - - $12ya^2 - 15y^2a + 3y^3$

(O) - - Reject $19y^2$ - - - $19y^2a - 19y^3$ G

(P) - - - - - - - - - - - - - - - $a - y)\ 4a^2 - 5ya + y^2(4a - y.$

$4a^2 - 4ya$

$- ya + y^2$

$- ya + y^2$

$\therefore$ The H. C. D. of (A) and (B) is $(b)(b + y)(a - y) = ab^2 + aby - b^2y - by^2$.

SCH.—It often occurs that one or more of the above steps are not required, especially the removing of a compound factor from the given polynomials.

2. Find the H. C. D., with respect to x, of $x^4 - 8x^3 + 21x^2 - 20x + 4$, and $2x^3 - 12x^2 + 21x - 10$.

OPERATION.

$2x^3 - 12x^2 + 21x - 10)\ x^4 - 8x^3 + 21x^2 - 20x + 4$

2

(C) - - - - - - - $2x^4 - 16x^3 + 42x^2 - 40x + 8(x - 2$

$2x^4 - 12x^3 + 21x^2 - 10x$

$- 4x^3 + 21x^2 - 30x + 8$

$- 4x^3 + 24x^2 - 42x + 20$

(D) Reject -3 - - - - - - - - $- 3x^2 + 12x - 12$

(E) - - - - - - - - - - - - - - $x^2 - 4x + 4$

$x^2 - 4x + 4)\ 2x^3 - 12x^2 + 21x - 10(2x - 4$

$2x^3 - 8x^2 + 8x$

$- 4x^2 + 13x - 10$

$- 4x^2 + 16x - 16$

(F) - Reject -3 - $- 3x + 6$ (E)

(G) - - - - - - - - - - - $x - 2)\ x^2 - 4x + 4(x - 2$

$x^2 - 2x$

$- 2x + 4$

$- 2x + 4$

Hence $x - 2$ is the H. C. D.

3. Find the H. C. D. of $2x^3 + 5 - 8x + x^2$, and $42x^2 + 30 - 72x$.

4. Find the H. C. D. of $2ax^2 + 2a + 4ax$, and $7b + 14bx + 7bx^3 + 14bx^2$.

5. Find the H. C. D. of $6a^2 + 7ax - 3x^2$, and $6a^2 + 11ax + 3x^2$.

6. Find the H. C. D. of $4a^3 - 4a^2 - ab^2 + b^2$, and $4a^2 + 2ab - 2b^2$.

7. Find the H. C. D. of $12x^4 - 24x^3y + 12x^2y^2$, and $8x^3y^2 - 24x^2y^3 + 24xy^4 - 8y^5$.

8. Find the H. C. D. of $52ax^3 - 24ax^4 - 44ax^2 - 12a + 8ax^5 + 60ax$, and $14a^2b + 60a^2bx^2 - 16a^2bx^3 + 2a^2bx^5 - 74a^2bx - 2a^2bx^4$.

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130. Prob.—*To find the H. C. D. of three or more polynomials.*

RULE.—FIND THE H. C. D. OF ANY TWO OF THE GIVEN POLYNOMIALS BY ONE OF THE FOREGOING METHODS, AND THEN FIND THE H. C. D. OF THIS H. C. D. AND ONE OF THE REMAINING POLYNOMIALS, AND THEN AGAIN COMPARE THIS LAST H. C. D. WITH ANOTHER OF THE POLYNOMIALS, AND FIND THEIR H. C. D. CONTINUE THIS PROCESS TILL ALL THE POLYNOMIALS HAVE BEEN USED.

DEM.—For brevity, call the several polynomials, A, B, C, D, etc. Let the H. C. D. of A and B be represented by P, whence P contains *all* the factors common to A and B. Finding the H. C. D. of P and C, let it be called P′. P′, therefore, contains *all* the common factors of P and C; and as P contains all that are common to A and B, P′ contains all that are common to A, B, and C. In like manner if P″ is the H. C. D. of P′ and D, it contains all the common factors of A, B, C, and D, etc. Q. E. D.

EXAMPLES.

1. Find the H. C. D. of $x^2 + 11x + 30$, $2x^2 + 21x + 54$, and $9x^3 + 53x^2 - 9x - 18$. *The H. C. D. is $x + 6$.*

2. What is the H. C. D. of $10x^5 + 10x^3y^2 + 20x^4y$, $2x^3 + 2y^3$, and $4y^4 + 12x^2y^2 + 4x^3y + 12xy^3$?

SECTION III.

LOWEST OR LEAST COMMON MULTIPLE.

131. DEF.—In speaking of decimal numbers, the term *Least Common Multiple* is correct, but not in speaking of literal numbers. For example, the numbers $(a+b)^2$ and (a^2-b^2) are both contained in $(a+b)^2 \times (a-b)$, and in any multiple of this product, as $m(a+b)^2(a-b)$. But whether $m(a+b)^2(a-b)$ is greater or less than $(a+b)^2(a-b)$ depends upon whether a is greater or less than b, and also whether m is greater or less than unity. In speaking of literal numbers, we should say *Lowest Common Multiple,* meaning the multiple of lowest degree with respect to some specified letter.

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132. Prob.—*To find the L. C. M. of two or more numbers.*

RULE.—TAKE THE LITERAL NUMBER OF THE HIGHEST DEGREE, OR THE LARGEST DECIMAL NUMBER, AND MULTIPLY IT BY ALL THE FACTORS FOUND IN THE NEXT LOWER WHICH ARE NOT IN IT. AGAIN, MULTIPLY THIS PRODUCT BY ALL THE FACTORS FOUND IN THE NEXT LOWER NUMBER AND NOT IN IT, AND SO CONTINUE TILL ALL THE NUMBERS ARE USED. THE PRODUCT THUS OBTAINED IS THE L. C. M.

DEM.—Let A, B, C, D, etc., represent any numbers arranged in the order of their degrees, or values. Now, as A is its own L. M., the L. C. M. of all the numbers must contain it as a factor. But, in order to contain B, the L. C. M. must contain all the factors of B. Hence, if there are any factors in B which are not found in A, these must be introduced. So, also, if C contains factors not found in A and B, they must be introduced, in order that the product may contain C, etc., etc. Now it is evident that the product so obtained, is the L. C. M. of the several numbers, since it contains all the factors of any one of them, and hence can be divided by any one of them, and if any factor were removed it would cease to be a multiple of some one or more of the numbers. Q. E. D.

1. Find the L. C. M. of (x^3-1), (x^2-1), and $(x+1)$.

SOLUTION.—The L. C. M. must contain x^3-1, and as it is its own L. M., if it contains all the factors of the other two, it is the required L. C. M. The factors of x^3-1 are $(x-1)(x^2+x+1)$. But this product does not contain the factors of (x^2-1), which are $(x+1)(x-1)$. Hence, we must introduce the factor $(x+1)$, giving $(x^3-1)(x+1)$, as the L. C. M. of x^3-1 and x^2-1. Now, as this product contains the third quantity, it is the L. C. M. of the three.

2. Find the L. C. M. of $(a+b)^2$, a^2-b^2, $(a-b)^2$, and $a^3+3a^2b+3ab^2+b^3$.

3. Find the L. C. M. of $(x^4 - 4)$, $(x^2 + 2)$, and $(x^2 - 2)$.

4. Find the L. C. M. of $(a^4 - 2a^2 + 1)$, $(1 + a)$, $(a - 1)$, and 4.

5. Find the L. C. M. of $3a^2b^2xy$, $57ax^3$, $87y^3$, and $9a^5b^{\frac{3}{2}}$.

6. Find the L. C. M. of $(1 - 18a + 81a^2)$, $(3a^{\frac{1}{2}} + 1)(1 - 3\sqrt{a})$, and $(27a^{\frac{3}{2}} - 9a - 3\sqrt{a} + 1)$.

Sch.—In applying this rule, if the common factors of the two numbers are not readily discerned, apply the method of finding the H. C. D., in order to discover them.

7. Find the L. C. M. of $x^3 - 2ax^2 + 4a^2x - 8a^3$, $x^3 + 2ax^2 + 4a^2x + 8a^3$, and $x^2 - 4a^2$.

Solution.—The L. C. M. of these numbers must contain $x^3 - 2ax^2 + 4a^2x - 8a^3$; and as it is its own L. M., if it contains all the factors of $x^3 + 2ax^2 + 4a^2x + 8a^3$, it is the L. C. M. of these two polynomials. But as the common factors of these numbers, if they have any, are not readily discerned, we apply the method of H. C. D., and find that $x^2 + 4a^2$ is the H. C. D. of the two. Since, then, $x^3 - 2ax^2 + 4a^2x - 8a^3$ contains the factor $x^2 + 4a^2$ of the second number, it is only necessary to introduce the other factor in order to have the L. C. M. of the two. Now, $(x^3 + 2ax^2 + 4a^2x + 8a^3) \div (x^2 + 4a^2) = x + 2a$. Hence, $(x^3 - 2ax^2 + 4a^2x - 8a^3)(x + 2a)$ or $x^4 - 16a^4$ is the L. C. M. of the first two numbers, since it contains all the factors of each, and no more. Now, to find whether $x^4 - 16a^4$ is a multiple of the remaining number, $x^2 - 4a^2$, or, if it is not, what factors must be introduced to make it so, we proceed in the same way as with the first two numbers. But our first step (or ***117***) shows us that $x^4 - 16a^4$ *is* a multiple of $x^2 - 4a^2$. $\therefore$ $x^4 - 16a^4$ is the L. C. M. of the three given numbers.

8. Find the L. C. M. of $x^2 - 3x - 70$ and $x^3 - 39x + 70$.

9. Find the L. C. M. of $x^2 + x - 2$, $x^2 - x - 6$, and $x^2 - 4x + 3$.

10. Find the L. C. M. of $a^3 - 4a^2b + 9ab^2 - 10b^3$ and $a^3 + 2a^2b - 3ab^2 + 20b^3$.

11. Find the L. C. M. of $x^3 - 9x^2 + 26x - 24$, $x^3 - 10x^2 + 31x - 30$, and $x^3 - 11x^2 + 38x - 40$.

12. Find the L. C. M. of $x^4 - 10x^2 + 9$, $x^4 + 10x^3 + 20x^2 - 10x - 21$, and $x^4 + 4x^3 - 22x^2 - 4x + 21$.

CHAPTER III.

FRACTIONS.

DEFINITIONS AND FUNDAMENTAL PRINCIPLES.

133. *A Fraction*, in the literal notation, is to be considered as an indicated operation in Division.

134. SCH.—In the literal notation it becomes impracticable to consider the denominator as indicating the number of equal parts into which unity is divided, and the numerator as indicating the number of those parts represented by the fraction, since the very genius of this notation requires that the letters be not restricted in their signification. Thus in $\frac{a}{b}$, it will not do to say, b represents the number of equal parts into which unity is divided, since the notation requires that whatever conception we take of these quantities should be sufficiently comprehensive to include all values. Hence b may be a mixed number. Now suppose $b = 4\frac{2}{3}$. It is absurd to speak of unity as divided into $4\frac{2}{3}$ equal parts.

135. COR. 1.—*Since numerator is dividend and denominator divisor, it follows from* (***91, 92, 93***) *that dividing or multiplying both terms of a fraction does not alter its value; that multiplying or dividing the numerator multiplies or divides the value of the fraction; and that multiplying or dividing the denominator divides or multiplies the fraction.*

136. COR. 2.—*A fraction is multiplied by its denominator by simply removing it.*

137. The terms *Integer* or *Entire Number, Mixed Number, Proper* and *Improper*, are applied to literal numbers, but not with strict propriety.

Whether $m + n$ is an integer, a mixed number, or a fraction, depends upon the values of m and n, which the genius of the literal notation requires to be understood as perfectly general, until some restriction is imposed.

As a matter of convenience, we adopt the following definitions:

138. A number not having the fractional *form* is said to have the ***Integral Form;*** as $m + n$, $2c^2d - 8a^{-2}x + 3x^5y^4$.

139. A polynomial having part of its terms in the fractional and part in the integral form, is called a ***Mixed Number.***

140. A Proper Fraction, in the literal notation, is an expression wholly in the fractional form, and which cannot be expressed in the integral form without negative exponents.

By calling such an expression a proper fraction, we do not assert anything with reference to its value as compared with unity. Thus $\frac{a}{b}$ is a proper fraction, though it may be greater or less than unity. It may also be written ab^{-1}.

141. An Improper Fraction is an expression in the fractional form, but which can be expressed in the integral or mixed form without the use of negative exponents.

142. A Simple Fraction is a single fraction with both terms in the integral form.

143. A Compound Fraction is two or more fractions connected by the word *of*.

This term is not generally applicable in the literal notation. Thus we may write $\frac{2}{3}$ of $\frac{3}{4}$ with propriety, but not $\frac{a}{b}$ of $\frac{m}{n}$, unless a and b are integral, so that the fraction $\frac{a}{b}$ may be considered as representing equal parts of unity, as $\frac{2}{3}$ does. If the word *of* is considered as simply an equivalent for $\times$, the notation is of course, always admissible. But it is scarcely a simple equivalent.

144. A Complex Fraction is a fraction having in one or both its terms an expression of the fractional form.

145. A fraction is in its *Lowest Terms* when there is no common integral factor in both its terms.

146. The Lowest Common Denominator is the number of lowest degree, which can form the denominator of several given fractions, giving fractions of the same values respectively, while the numerators retain the integral form.

147. Reduction, in mathematics, is changing the form of an expression without changing its value.

Reductions.

148. There are five principal reductions required in operating with fractions, viz.: *To Lowest Terms,—From Improper Fractions to Integral or Mixed Forms,—From Integral or Mixed Forms to Improper Fractions,—To Forms having a Common Denominator,*—and from the *Complex to the Simple Form.*

149. Prob. 1.—*To reduce a fraction to its lowest terms.*

RULE.—Reject all common factors from both terms; or divide both terms by their H. C. D.

Dem.—Since the numerator is the dividend and the denominator the divisor, rejecting the same factors from each does not alter the value of the fraction (***91***). Having rejected *all* the common factors, or, what is the same thing, the H. C. D. (which contains all the common factors), the fraction is in its lowest terms (***145***).

Sch. 1.—Since the H. C. D. is the product of all the common factors (***109***), the above process is equivalent to dividing both terms of the fraction by their H. C. D. Whenever the common factors of the terms are not readily discernible, the process for finding their H. C. D. (***129***) may be resorted to.

Sch. 2.—The opposite process is sometimes serviceable, viz.: the introduction of a factor into both terms of a fraction, which will give it a more convenient form. It requires no special ingenuity to solve such problems, since, if the factor does not readily appear, it can be found by dividing a term of one fraction by the corresponding term of the other.

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150. Prob. 2.—*To reduce a fraction from an improper to an integral or mixed form.*

RULE.—Perform the division indicated (***133***).

151. Cor.—*By means of negative indices (exponents) any fraction can be expressed in the integral form.*

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152. Prob. 3.—*To reduce numbers from the integral or mixed to the fractional form.*

RULE.—Multiply the integral part by the given denominator, and annexing the numerator of the fractional part, if any, write the sum over the given denominator.

DEM.—In the case of a number in the integral form, the process consists of multiplying the given number by the given denominator and indicating the division of the product by the same number, and hence is equivalent to multiplying and dividing by the same quantity, which does not change the value of the number. The same is true as far as relates to the integral part of a mixed form, after which the two fractional parts are to be added together. As they have the same divisors, the dividends can be added upon the principle that the sum of the quotients equals the quotient of the sum (***94***).

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153. Prob. 4.*—To reduce fractions having different denominators to equivalent fractions having a common denominator.*

RULE.—MULTIPLY BOTH TERMS OF EACH FRACTION BY THE DENOMINATORS OF ALL THE OTHER FRACTIONS.

DEM.—This gives a common denominator, because each denominator is the product of all the denominators of the several fractions. The value of any one of the fractions is not changed, because both numerator and denominator are multiplied by the same number (***135***).

154. COR.—*To reduce fractions to equivalent ones having the Lowest Common Denominator, find the L. C. M. of all the denominators for the new denominator. Then multiply both terms of each fraction by the quotient of that L. C. M. divided by the denominator of that fraction.*

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155. Prob. 5.*—To reduce complex fractions to the form of simple fractions.*

RULE.—MULTIPLY NUMERATOR AND DENOMINATOR OF THE COMPLEX FRACTION BY THE PRODUCT OF ALL THE DENOMINATORS OF THE PARTIAL FRACTIONS FOUND IN THEM; OR, MULTIPLY BY THE L. C. M. OF THE DENOMINATORS OF THE PARTIAL FRACTIONS.*

DEM.—This process removes the partial denominators, since each fraction is multiplied by its own denominator, at least, and this is done by dropping the denominator. It does not alter the value of the fraction, since it is multiplying dividend and divisor by the same quantity.

ADDITION.

156. Prob.*—To add fractions.*

RULE.—REDUCE THEM TO FORMS HAVING A COMMON DENOMINATOR, IF THEY HAVE NOT SUCH FORMS, AND THEN ADD THE NUMERATORS, AND WRITE THE SUM OVER THE COMMON DENOMINATOR.

* The pupil is supposed to have obtained sufficient knowledge of fractions in common arithmetic to perform these operations.

DEM.—The reduction of the several fractions to forms having a common denominator, if they have not such forms, does not alter their values (**135**), and hence does not alter the sum. Then, when they have a common denominator (divisor), the sum of the several quotients is equal to the quotient of the sum of the several dividends divided by the common divisor, or denominator (**94**).

157. COR.—*Expressions in the mixed form may either be reduced to the improper form and then added, or the integral parts may be added into one sum, and the fractional into another, and these results added.*

SUBTRACTION.

158. Prob.—*To subtract fractions.*

RULE.—REDUCE THE FRACTIONS TO FORMS HAVING A COMMON DENOMINATOR, IF THEY HAVE NOT SUCH FORMS, AND SUBTRACT THE NUMERATOR OF THE SUBTRAHEND FROM THE NUMERATOR OF THE MINUEND, AND PLACE THE REMAINDER OVER THE COMMON DENOMINATOR.

DEM.—The value of the fractions not being altered by the reduction, their difference is not altered. After this reduction, we have the difference of two quotients arising from dividing two numbers (the numerators) by the same divisor (the common denominator). But this is the same as the quotient arising from dividing the difference between the numbers by the common divisor (**95**).

159. COR.—*Mixed numbers may be subtracted by annexing the subtrahend with its signs changed, to the minuend, and then combining the terms as much as may be desired. The reason for the change of signs is the same as in whole numbers* (**71**).

MULTIPLICATION.

160. Prob. 1.—*To multiply a fraction by an integer.*

RULE.—MULTIPLY THE NUMERATOR OR DIVIDE THE DENOMINATOR.

DEM.—Since numerator is dividend and denominator divisor, and the value of the fraction is the quotient, this rule is a direct consequence of (**92, 93**).

161. Prob. 2.—*To multiply by a fraction.*

RULE.—MULTIPLY BY THE NUMERATOR AND DIVIDE BY THE DENOMINATOR.*

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* It is assumed that the pupil knows how to divide a fraction by an integer, from his study of arithmetic. Nevertheless the problem will be introduced hereafter for the purpose of familiarizing the pupil with the literal operations.

DEM.—Let it be required to multiply m, which is either an integer or a fraction, by $\frac{a}{b}$.

1st. Suppose a and b are both integers. Multiplying m by a gives a product b times too large, since we were to multiply by only a bth part of a; hence we divide the product, am, by b, and have $\frac{am}{b}$.

2d. When either a or b, or both, are fractions. Let c be the factor by which numerator and denominator of $\frac{a}{b}$ must be multiplied to make $\frac{a}{b}$ a simple fraction (***155***). Then will $\frac{ac}{bc}$ be a simple fraction, *i. e.*, ac and bc are each integral; and the multiplication is effected as in Case 1st, giving $\frac{acm}{bc}$. This reduced by dividing both terms by c gives $\frac{am}{b}$. Hence we see that in *any* case, to multiply by a fraction, we have only to multiply the multiplicand by the numerator of the multiplier, and divide this product by the denominator. It is also to be observed that this reasoning applies equally well whether the *multiplicand* is integral or fractional.

162. COR.—*To multiply mixed numbers, first reduce them to improper fractions.*

DIVISION.

163. Prob. 1.*—To divide a fraction by an integer.*

RULE.—DIVIDE THE NUMERATOR OR MULTIPLY THE DENOMINATOR.

DEM.—Since numerator is dividend, and denominator divisor, and the value of the fraction the quotient, this rule is a direct consequence of (***92, 93***).

164. Prob. 2.*—To divide by a fraction.*

RULE.—DIVIDE BY THE NUMERATOR AND MULTIPLY THE QUOTIENT BY THE DENOMINATOR. OR, WHAT IS THE SAME THING, INVERT THE TERMS OF THE DIVISOR AND PROCEED AS IN MULTIPLICATION.

DEM.—The correctness of the first process appears from the fact that division is the reverse of multiplication, and, hence, as we multiply by the numerator and divide by the denominator in order to multiply by a fraction, to divide by one we must *divide* by the numerator and *multiply* by the denominator.

The process of inverting the divisor and then multiplying by it is seen to be the same as the other, since this multiplies the dividend by the denominator of the divisor and divides by the numerator.

Again, this process may be demonstrated thus: Inverting the divisor shows how many times it is contained in 1. Then if the given divisor is contained so many times in 1, it will be contained in 5, 5 times as many times; in $\frac{2}{3}$, $\frac{2}{3}$ as many times; in ax^2, ax^2 times as many times; or in any dividend as many times the number of times it is contained in 1, as is expressed by that dividend, whether it be integral, fractional, or mixed. (The author prefers this demonstration.)

SCH. 1.—Since to multiply one fraction by another we may multiply the numerators together for the numerator and the denominators for the denominator, and since division is the reverse, we may perform division by dividing the numerator of the dividend by the numerator of the divisor, and the denominator of the dividend by the denominator of the divisor.

This method will coincide with the others when *they* are worked by performing the operations by division as far as practicable, and *this* is worked by performing the multiplications equivalent to the divisions when the latter are not practicable.

165. COR.—*The reciprocal of a quantity being* 1 *divided by that quantity, the reciprocal of a fraction is the fraction inverted.*

GENERAL SCHOLIUM.—In both multiplication and division of fractions, or by fractions, all operations which can be performed by *dividing* should be so performed, in order that the result may be in its lowest terms.

SIGNS OF A FRACTION.

166. In considering the signs of a fraction, we have to notice *three things*, viz.: the sign of the numerator, the sign of the denominator, and the sign before the fraction as a whole. This latter sign does not belong to either the numerator or denominator separately, but to the whole expression.

Thus, in the expression $-\frac{4a-5cd}{2x+4y^2}$, in the numerator the sign of $4a$ is $+$, and of $5cd$ is $-$. In the denominator, the sign of $2x$ is $+$, and of $4y^2$ $+$ also. The sign of the fraction is $-$. These are the signs of operation (***50***).

167. The essential character of a fraction, as *positive* or *negative*, can be determined only when the essential character of all the numbers entering into it is known. It may then be determined by principles already given (***78, 97***).

Examples.

1. Reduce the following fractions to their lowest terms:

$$\frac{770a^3by^2}{1210a^3b^2y^4},\ \frac{25ax^3}{55a^3x},\ \frac{72a^5b^{\frac{3}{2}}c^{2n}}{24ab^{\frac{1}{2}}c^n},\ \frac{x+1}{1-x^2},\ \frac{3m^2n-3n^3}{12m^2n^2+24mn^3+12n^4},$$

$$\frac{x^{-4}-y^{-4}}{x^{-1}-y^{-1}},\ \frac{a^2b-1}{1+a\sqrt{b}},\ \frac{x^2-3x-4}{x^2-4x-5},\ \frac{3a+3ag}{4b^{\frac{3}{2}}-4b^{\frac{3}{2}}g^2},\ \frac{3x^2+12x+9}{x^5+5x^3+6},$$

$$\frac{6x^2-3x-45}{6x^2+19x+10},\ \frac{(1+x)^2}{(1-x^2)^2},\ \frac{a^{p-q+1}b^{r-s}c^{\frac{2}{3}}}{a^{q-p+1}b^{s+r}c^{\frac{3}{2}}},\ \frac{2x^3y^3+2}{x^6y^6-1},\ \frac{3x^3+2x^2-8x}{9x^3-12x^2-36x+48},$$

$$\frac{2x^4-x^3-9x^2+13x-5}{7x^3-19x^2+17x-5},\ \frac{2ab^3+ab^2-8ab+5a}{7b^3-12b^2+5b},\ \frac{x^3-8x^2+21x-18}{3x^3-16x^2+21x},$$

$$\frac{16x^4-53x^2+45x+6}{8x^4-30x^3+31x^2-12}.$$

2. What factor will change $\dfrac{a^3-a^2b-ab^2+b^3}{a-b}$ to $\dfrac{(a^2-b^2)^2}{a^2-b^2}$? $\dfrac{a^4+a^3x+a^2x^2+ax^3+x^4}{a+x}$ to $\dfrac{a^5-x^5}{a^2-x^2}$? $\dfrac{x^3+\frac{3}{4}x^2+\frac{3}{4}x+1}{\frac{1}{4}x^2-\frac{1}{4}}$ to $\dfrac{4x^2-x+4}{x-1}$? $\dfrac{6p^4-12pq^3-6p^3q+12q^4}{p^2-q^2}$ to $\dfrac{6[p^3-2q^3]}{p+q}$?

3. Reduce $\dfrac{1}{1-x}$, $\dfrac{x^4+14x^2+27}{x^2+4}$, $\dfrac{m^2+n^2+2mn-x-y}{m+n}$, $\dfrac{1-a^{10}}{a+1}$, $\dfrac{2}{2-x}$, $\dfrac{a^4+4a^3x+6a^2x^2+4ax^3+x^4}{a^3+3a^2x+3ax^2+x^3}$, $\dfrac{x^n}{1-x^{-n}}$, and $\dfrac{a^{2n+1}}{a^{2n+1}+na}$ to integral or mixed forms.

4. Express $\dfrac{3a^2by^3}{7a^3by^2}$, $\dfrac{(m+n)^3}{(n+m)^{-m}}$, $\dfrac{c^2z^{a+2}}{c^{-3}d^nz^{2-a}}$, and $\dfrac{x^{-n}(a+b)^3}{x^{3-n}(a-b)^3}$ in the integral form.

5. Reduce the following mixed to fractional forms: $1+\dfrac{x}{1-x}$, $1+7a+6ab-\dfrac{14a^2b+12a^2b^2-3ab}{2ab}$, $1+2x-\dfrac{4x-4}{5x}$, $a-b+\dfrac{3a^2b+2ab^2-b^3}{a^2-b^2}$, $1+\dfrac{b^2+c^2-a^2}{2bc}$, and $1-x-\dfrac{1-x}{1+x}$.

6. Reduce $\frac{3a}{5b}$, $\frac{4x}{7y^2}$, $\frac{ay}{n^2}$, $\frac{7y}{3x}$, and $\frac{4n^2}{4m^2}$ to forms having a C. D.

7. Reduce $\frac{a+x}{x}$, $\frac{a-x}{a}$, and $\frac{a^2-x^2}{a^2+x^2}$ to forms having a C. D.

8. Reduce $\frac{a-x}{x}$, $\frac{a+x}{x(a^2-x^2)}$, $\frac{a-x}{a+x}$, and $\frac{1}{a-x}$ to equivalent fractions having the L. C. D.

9. Reduce $\frac{1}{x-y}$, $\frac{x}{(x-y)^2}$, and $\frac{x^2}{(x-y)^3}$ to equivalent fractions having the L. C. D.

10. Reduce $\frac{1+x^3}{(1+x)^3}$ and $\frac{(1-x)^3}{1-x^3}$ to forms having the L. C. D.

11. Reduce $\frac{m^2}{n^2}$, $\frac{n^2}{m^2}$, $\frac{m^2-n^2}{m^2+n^2}$, and $\frac{m^2+n^2}{m^2-n^2}$ to forms having the L. C. D.

12. Reduce $\frac{2}{3x}$, $\frac{3x}{3x+4}$, and $\frac{2+3x}{9x^2-16}$ to forms having the L. C. D.

13. Reduce the following complex fractions to simple ones:

$$\frac{1+\frac{b}{a}}{bc}; \quad \frac{\frac{2}{3}a^2-\frac{1}{4}a^2}{\frac{5}{6}b^2y^2+\frac{7}{24}c^2x^2}; \quad \frac{\frac{x}{3}}{\frac{3}{x}+y^2}; \quad \frac{\frac{a}{b}-\frac{c}{d}}{\frac{e}{f}+\frac{g}{h}}; \quad \frac{\frac{m^2+n^2}{n}-m}{n^{-1}-m^{-1}};$$

$$\frac{a}{b+\frac{c}{d+\frac{e}{f}}}.$$

14. Add $\frac{a}{3}$, $\frac{a}{5}$, $\frac{a}{7}$, and $\frac{a}{11}$; $\frac{x-7}{7}$ and $\frac{7-x}{7^2}$; $\frac{1}{1-x^2}$ and $\frac{1}{1+x^2}$; $\frac{a+b}{2}$ and $\frac{a-b-2c}{2}$; $\frac{2}{x^3+x^2+x+1}$ and $\frac{3}{x^3-x^2+x-1}$; $\frac{a-b}{ab}$, $\frac{c-a}{ac}$, and $\frac{b-c}{bc}$; $\frac{1}{x+3}$, $\frac{x+1}{x^2-3x+9}$, and $\frac{x^2+x+1}{x^3+27}$; $\frac{1}{a-b}$, $\frac{1}{a+b}$, and $-\frac{2a}{a^2-b^2}$.

15. Add $\frac{x}{a^2 - ax + x^2}$, $\frac{1}{a + x}$, and $\frac{a^2}{a^3 + x^3}$; $a - \left(\frac{3x^2}{b} + 4a^{\frac{2}{3}}x^{\frac{1}{5}}\right)$ and $b + \frac{2ax}{c} + 4a^{\frac{2}{3}}x^{\frac{1}{5}}$.

16. Add $\frac{3}{a}$, $-\frac{a+1}{b-1}$, and $-\frac{a-1}{b+1}$; $\frac{1+x}{1+x+x^2}$ and $\frac{1-x}{1-x+x^2}$; $\frac{x+3}{x+4}$, $\frac{x-4}{x-3}$, and $\frac{x+5}{x+7}$.

17. Add $\frac{1}{(a+c)(a+d)}$ and $\frac{1}{(a+c)(a+e)}$; $\frac{p}{3my^2 - x}$ and $\frac{y - 6mpy^2}{(3my^2 - x)^2}$; $\frac{x}{y}$, $\frac{y}{x+y}$, and $\frac{x^2}{x^2 + xy}$.

18. Add $\frac{2}{a-b}$, $\frac{2}{b-c}$, $\frac{2}{c-a}$, and $\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{(a-b)(b-c)(c-a)}$.

19. From $\frac{x}{x-3}$ take $\frac{x}{x+3}$; from $\frac{(a+b)^2}{ab}$ take $\frac{(a-b)^2}{ab}$; from $\frac{1}{x-7}$ take $\frac{1}{x-3}$.

20. From $7x - \frac{3a - 2b}{3}$ take $x - \frac{4a - b}{2}$; from $\frac{(a^2 + b^2)^2}{ab(a-b)^2}$ take $\frac{a}{b} + \frac{b}{a} + 2$.

21. From $\frac{5}{2(x+1)} - \frac{1}{10(x-1)}$ take $\frac{24}{5(2x+3)}$.

22. Multiply $\frac{4x + 12}{7}$ by $\frac{14a}{3x + 9}$; $\frac{x^2 - b^2}{ax + ab}$ by $\frac{x^2 + b^2}{x - b}$; $\frac{a-b}{a+b}$ by $\frac{a+b}{a-b}$; $\frac{a-x}{b+y}$ by $\frac{a+y}{b-x}$; $\frac{b}{a} + b$ by $\frac{a}{b} + \frac{1}{b}$; $\frac{x^2 + 2x - 3}{x^2 + 5x + 6}$ by $\frac{4x^2 - 12x - 40}{3x^2 - 18x + 15}$.

23. Multiply $\dfrac{a^{-1}+b}{4c}$ by $4c$; $-\frac{1}{2}x^{-1}$ by $\frac{1}{4}x^2$; $-\dfrac{2a^{-2}}{5b^{-3}}$ by $-\dfrac{10b^{-2}}{27a^2}$; $\dfrac{1+x}{1-x}$ by $1-2x+x^2$; $\dfrac{a^{\frac{2}{3}}}{b^{n+1}}$ by $-\dfrac{b^{1-n}}{a^{-\frac{2}{3}}}$; $\dfrac{x^{1+m}}{y^{1-n}}$ by $\dfrac{y^{1+n}}{x^{m+1}}$.

24. Multiply $\dfrac{x^{\frac{2}{3}}-2x^{\frac{1}{3}}+1}{y^{\frac{3}{4}}+y^{\frac{1}{2}}+y^{\frac{1}{4}}+1}$ by $\dfrac{x^{\frac{1}{3}}-1}{y^{\frac{1}{4}}-1}$; $\dfrac{a^m+b^p-2c^n}{a^{m-n}b^{n-p}}$ by the reciprocal of $\dfrac{2a^m-3b}{a^{n-m}b^{p-n}c}$; $\left(l+\dfrac{1}{l}\right)\left(l^2+\dfrac{1}{l^2}\right)$ by $\left(l-\dfrac{1}{l}\right)$; $\dfrac{1}{a}-\dfrac{1}{b}+\dfrac{1}{c}$ by $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$; $1+\frac{1}{2}x+\frac{1}{4}x^2$ by $1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{16}x^3$; $\left[\dfrac{x^2}{x^2-y^2}-\dfrac{y^2}{x^2+y^2}\right]$ by $\dfrac{(x^2-y^2)^2}{(x^2-y^2)^2+(x^2+y^2)^2}$; $\dfrac{a^2-1}{(a+1)^2}$ by $\dfrac{a^6-1}{(a^2-a)^2}$.

25. Multiply together $\dfrac{1-x^2}{1+y}$, $\dfrac{1-y^2}{x+x^2}$, and $1+\dfrac{x}{1-x}$.

26. Multiply x^2-x+1 by $x^{-2}+x^{-1}+1$; $1-\dfrac{a-b}{a+b}$ by $2+\dfrac{2b}{a-b}$.

27. Divide $\dfrac{7a^2}{2}$ by $\dfrac{3b^2}{13}$; $\dfrac{1}{m^2n^2}$ by m^2n^2; $\dfrac{1}{m^2n^2}$ by $m^{-2}n^{-2}$; $\dfrac{my^n}{4ax}$ by $\dfrac{my^n}{ax}$.

28. Divide $\dfrac{7a^3b^{\frac{4}{5}}x^{\frac{1}{3}}}{11mn^{\frac{2}{3}}y^{\frac{3}{4}}}$ by $11m^3n^{-\frac{2}{3}}y^{\frac{4}{5}}x^{\frac{1}{3}}$; $\dfrac{1}{1-9a^9}$ by $1+9a^9$; $\dfrac{1}{x^{m^n}}$ by x^{m^n}.

29. Divide $\dfrac{1-a^6}{1+a}$ by $1-a$; $\dfrac{a^2-b^2}{a+2b}$ by $\dfrac{a-b}{3a+6b}$; $\left(\dfrac{x}{a}-\dfrac{a}{x}\right)$ by $\dfrac{(a+x)^2}{ax}$; $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right)$ by $\left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right)$; $\left(\dfrac{c-b}{c+b}-\dfrac{c^3-b^3}{c^3+b^3}\right)$ by $\left(\dfrac{c+b}{c-b}+\dfrac{c^2+b^2}{c^2-b^2}\right)$.

30. Divide m^4-n^{-4} by $m+\dfrac{1}{n}$; $\dfrac{4(a^2-ab)}{b(a+b)^2}$ by $\dfrac{6ab}{a^2-b^2}$; $a^2-b^2-c^2-2bc$ by $\dfrac{a+b+c}{a+b-c}$; $\left(\dfrac{x+2y}{x+y}+\dfrac{x}{y}\right)$ by $\left(\dfrac{x+2y}{y}-\dfrac{x}{x+y}\right)$.

31. Divide $\dfrac{1-\dfrac{2xy}{(x+y)^2}}{1+\dfrac{2xy}{(x-y)^2}}$ by $\left(\dfrac{1-\dfrac{y}{x}}{1+\dfrac{y}{x}}\right)^2$.

32. Divide $\dfrac{a+(b^2a)^{\frac{1}{3}}-(a^2b)^{\frac{1}{3}}}{a+b}$ by $\dfrac{a^{\frac{1}{3}}}{a^{\frac{1}{3}}+b^{\frac{1}{3}}}$.

33. Divide $\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}-3a^{-1}b^{-1}c^{-1}$ by $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.

34. Free $\dfrac{a^{-2}+b^{-5}}{c^{-4}-d^{-7}}$, $\dfrac{x^{-\frac{2}{3}}y^{-n}}{a^{-\frac{2}{3}}b^{-12}}$, $\dfrac{x^{-\frac{2}{3}}+y^{-n}}{a^{-\frac{2}{3}}b^{-12}}$, $\dfrac{(x-y)^{-4}}{x^{-4}+y^{-4}}$, and $a^{-2}b + b^{-2}a$ of negative exponents.

35. What is the reciprocal of $\dfrac{c+d}{cd-1}$, $\left(\dfrac{5x^3}{27}\right)^{\frac{1}{3}}$, $\sqrt{\dfrac{m+n}{m-n}}$, and $\dfrac{(m+n)^{-6}}{(m-n)^{-6}}$?

36. Is the fraction $-\dfrac{4a^2-3mx}{2x^3+4y^2}$ essentially positive, or negative, when a, m, x, and y are each negative?

Solution.—Since $(-a)^2 = a^2$, $4a^2$ is essentially positive. Since $(-m)(-x) = mx$, the term $3mx$, *in itself, is positive*, and the numerator becomes $4a^2 - (+3mx)$, or $4a^2 - 3mx$ (72). Now, whether $4a^2 - 3mx$ gives a + or a − result, depends upon the numerical values of a, m, and x. If $4a^2 > 3mx$, $4a^2 - 3mx$ is +; but, if $4a^2 < 3mx$, $4a^2 - 3mx$ is −. Again, since $(-x)^3 = -x^3$, the first term of the denominator, $2x^3$, is essentially negative. And since $(-y)^2 = y^2$, the term $4y^2$ is essentially positive and the denominator becomes $-2x^3 + (+4y^2)$, or $-2x^3 + 4y^2$. Whether this is + or −, depends upon the relative values of x and y. If we suppose $4a^2 > 3mx$ the numerator becomes +, and if $2x^3$ be greater than $4y^2$ the denominator becomes −, and we have $-\dfrac{+}{-}$, which gives a positive result.

37. What is the essential sign of $-\dfrac{3a^2xy-7b^2}{abxy-4}$, when $a=-1$, $b=2$, $x=-3$, and $y=-4$?

38. What is the essential sign of $\dfrac{3am^3-7b^{\frac{1}{3}}y}{-4bm}$, when $a=-3$, $b=-8$, $m=-1$, and $y=1$?

39. What is the essential sign of $-\dfrac{-2a^{\frac{1}{5}}x^2 - 3a^{\frac{3}{5}}b}{-am^{\frac{2}{3}} - 5b^2m^{\frac{1}{3}}}$, when $a = -32$, $b = -2$, $m = -8$, and $x = -2$?

40. Simplify $\dfrac{1}{x + \dfrac{1}{y + \dfrac{1}{z}}} \div \dfrac{1}{x + \dfrac{1}{y}} - \dfrac{1}{y(xyz + x + z)}$,

$$\frac{\dfrac{1}{a-x} - \dfrac{1}{a-y} + \dfrac{x}{(a-x)^2} - \dfrac{y}{(a-y)^2}}{\dfrac{1}{(a-y)(a-x)^2} - \dfrac{1}{(a-y)^2(a-x)}},$$

$$\frac{\dfrac{3}{abc}}{\dfrac{1}{bc} + \dfrac{1}{ca} - \dfrac{1}{ab}} - \frac{3 - a - b + c}{a + b - c}, \text{ and}$$

$$\frac{a + \dfrac{b}{1 + \dfrac{a}{b}}}{a - \dfrac{b}{1 - \dfrac{a}{b}}}(a^6 - b^6).$$

CHAPTER IV.

POWERS AND ROOTS.

SECTION I.

INVOLUTION.

DEFINITIONS.

168. *A Power* is a *product* arising from multiplying a number by itself. ***The Degree*** of the power is indicated by the number of factors taken.

SCH.—It will be seen that a power is a species of composite number in which the component factors are equal.

169. *A Root* is one of the equal factors into which a number is conceived to be resolved. ***The Degree*** of the root is indicated by the number of required factors.

170. *An Exponent* or ***Index*** is a number written a little to the right and above another number, and

1st. *If a Positive Integer*, it indicates a *Power* of the number;

2d. *If a Positive Fraction,* the numerator indicates a *Power,* and the denominator a *Root* of the number;

3d. *If a Negative Integer or Fraction,* it indicates the *Reciprocal* of what it would signify if positive.

SCH.—It is obviously incorrect to read $4^{\frac{2}{5}}$, "the $\frac{2}{5}$ *power* of 4." There is no such thing as a 2-fifths power, as will be seen by considering the definition of a power. Read $4^{\frac{2}{5}}$, "4 exponent $\frac{2}{5}$;" also $a^{\frac{m}{n}}$, "a exponent $\frac{m}{n}$;" $a^{-\frac{m}{n}}$, "a exponent $-\frac{m}{n}$." These are abbreviated forms for, "a with an exponent $-\frac{m}{n}$," etc. In this way any exponent, however complicated, is read without difficulty.

171. *A Radical Number* is an indicated root of a number. If the root can be extracted exactly, the quantity is called *Rational;* if the root cannot be extracted exactly, the expression is called *Irrational,* or *Surd.*

172. A *Root* is *indicated* either by the denominator of a fractional exponent, or by the ***Radical Sign,*** $\sqrt{\ }$. This sign used alone signifies square root. Any other root is indicated by writing its index in the opening of the v part of the sign.

173. *An Imaginary Quantity* is an indicated *even* root of a negative quantity, and is so called because no number, in the ordinary sense, can be found, which, taken an *even* number of times as a factor, produces a negative quantity.

Thus $\sqrt{-4}$ is imaginary, because we cannot find any factor, in the ordinary sense, which multiplied by itself once produces -4. Neither $+2$ nor -2 produces -4 when squared. For a like reason $\sqrt{-3a^2}$, $\sqrt{-5x}$, or $\sqrt[4]{-140xy^2}$ are imaginaries.

174. All quantities not imaginary are called *Real.*

175. *Similar Radicals* are like roots of like quantities.

Thus $4\sqrt{5a}$, $3x\sqrt{5a}$, and $(a^2-x^2)\sqrt{5a}$ are similar radicals.

176. *To Rationalize* an expression is to free it from radicals.

177. *To affect a number with an Exponent* is to perform upon it the operations indicated by that exponent.

178. *Involution* is the process of raising numbers to required powers.

179. *Evolution* is the process of extracting roots of numbers.

180. *Calculus of Radicals* treats of the processes of reducing, adding, subtracting, or performing any of the common arithmetical operations upon radical quantities.

INVOLUTION.

181. *Prob. 1.*—*To raise a number to any required power.*

RULE.—MULTIPLY THE NUMBER BY ITSELF AS MANY TIMES, LESS ONE, AS THERE ARE UNITS IN THE DEGREE OF THE POWER.

182. COR.—*Since any number of positive factors gives a positive product, all powers of positive monomials are positive. Again,* *since an* EVEN *number of negative factors gives a* POSITIVE *product, and an* ODD *number gives a* NEGATIVE *product, it follows that even powers of negative numbers are positive, and odd powers negative.*

183. *Prob. 2.*—*To affect a monomial with any exponent.*

RULE.—PERFORM UPON THE COEFFICIENT THE OPERATIONS INDICATED BY THE EXPONENT, AND MULTIPLY THE EXPONENTS OF THE LETTERS BY THE GIVEN EXPONENT.

DEM.—1st. *When the exponent by which the monomial is to be affected is a positive integer.* Let it be required to affect $4a^m b^{\frac{n}{r}} x^{-s}$ with the exponent p; or in other words raise it to the pth power, p being a positive integer. The pth power of $4a^m b^{\frac{n}{r}} x^{-s}$ is $4a^m b^{\frac{n}{r}} x^{-s} \times 4a^m b^{\frac{n}{r}} x^{-s} \times 4a^m b^{\frac{n}{r}} x^{-s}$ - - - - - to p factors. But as the order of the arrangement of the factors does not affect the product (**77**), this product may be considered as, p factors each 4, into p factors each a^m, into p factors each $b^{\frac{n}{r}}$, into p factors each x^{-s}. Now p factors each 4 give 4^p by definition. p factors each a^m are expressed a^{pm}, since a^m is m factors each a, and p factors containing m factors each, make in the whole pm factors, or a^{pm}. Again, p factors each $b^{\frac{n}{r}}$ are expressed $b^{\frac{pn}{r}}$, since $b^{\frac{n}{r}}$ is n factors each $b^{\frac{1}{r}}$, and p factors, containing n *factors* each, are pn factors each $b^{\frac{1}{r}}$, or $b^{\frac{pn}{r}}$. And since $x^{-s}=\frac{1}{x^s}$, p factors, or $\frac{1}{x^s}\times\frac{1}{x^s}\times\frac{1}{x^s}$ - - - to p factors make $\frac{1}{x^{sp}}$, as fractions are multiplied by multiplying numerators together for a new numerator and denominators for a new denominator, and $x^s \times x^s \times x^s$ - - - to p factors are x^{ps}. But $\frac{1}{x^{ps}} = x^{-ps}$. Hence collecting the factors we find that $(4a^m b^{\frac{n}{r}} x^{-s})^p = 4^p a^{pm} b^{\frac{pn}{r}} x^{-ps}$. Q. E. D.

2d. *When the exponent is a positive fraction.* Let it be required to affect $4a^m b^{\frac{n}{r}} x^{-s}$, with the exponent $\frac{p}{q}$. This means that $4a^m b^{\frac{n}{r}} x^{-s}$ is to be resolved into q equal factors and p of them taken. Now, if we separate each of the factors of $4a^m b^{\frac{n}{r}} x^{-s}$ into q equal factors, and then take p of each of these, we shall have done what is signified by the exponent $\frac{p}{q}$.

By definition, $4^{\frac{1}{q}}$ represents *one* of the q equal factors of 4.

To obtain *one* of the q equal factors of a^m, we take *one* of the q equal factors of a from each of the m factors represented. But *one* of the q equal factors of a is represented by $a^{\frac{1}{q}}$, and m of these is $a^{\frac{m}{q}}$ by definition.

To separate $b^{\frac{n}{r}}$ into q equal factors, we notice that $b^{\frac{n}{r}}$ is n of the r equal factors of b. Now, if we resolve each of these r factors into q equal factors, b is resolved into rq equal factors; doing the same with each of the n factors represented, and taking *one* from each set, we have b resolved into rq equal factors and n of them taken; that is $b^{\frac{n}{rq}}$ is *one* of the q equal factors of $b^{\frac{n}{r}}$.

To resolve $x^{-s} = \frac{1}{x^s}$ into q equal factors, we consider that a fraction is resolved by resolving its numerator and denominator separately. But one of the q equal factors of 1 is 1; and *one* of the q equal factors of x^s is $x^{\frac{s}{q}}$ as seen in the resolution of a^m. Hence *one* of the q equal factors of x^{-s} or $\frac{1}{x^s}$ is $\frac{1}{x^{\frac{s}{q}}} = x^{-\frac{s}{q}}$.

Collecting these factors we find that *one* of the q equal factors of $4a^mb^{\frac{n}{r}}x^{-s}$ is $4^{\frac{1}{q}}a^{\frac{m}{q}}b^{\frac{n}{qr}}x^{-\frac{s}{q}}$. And finally p of these being obtained according to Case 1st, gives $4^{\frac{p}{q}}a^{\frac{pm}{q}}b^{\frac{pn}{qr}}x^{-\frac{ps}{q}}$, as the expression for $4a^mb^{\frac{n}{r}}x^{-s}$ affected with the exponent $\frac{p}{q}$; which result agrees with the enunciation of the rule.

3d. *When the exponent is negative and either integral or fractional.* Let it be required to affect $4a^mb^{\frac{n}{r}}x^{-s}$ with the exponent $-t$. This by the definition of negative exponents, signifies that we are to take the reciprocal of what the expression would be if t were positive. But $4a^mb^{\frac{n}{r}}x^{-s}$ affected with the exponent t (positive) is $4^ta^{tm}b^{\frac{tn}{r}}x^{-ts}$ by the preceding cases, whether t is integral or fractional. The reciprocal of this is $\frac{1}{4^ta^{tm}b^{\frac{tn}{r}}x^{-ts}}$. But since these factors can be transferred to the numerator by changing the signs of their exponents, we have $4^{-t}a^{-tm}b^{-\frac{tn}{r}}x^{ts}$, as the result of affecting $4a^mb^{\frac{n}{r}}x^{-s}$ with the exponent $-t$, which result agrees with the enunciation of the rule.

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184. Prob. 3.—*To expand a binomial affected with any exponent.*

RULE.—THIS RULE IS BEST STATED IN A FORMULA. THUS, LET a, b, AND m BE ANY NUMBERS WHATEVER, POSITIVE OR

NEGATIVE, INTEGRAL OR FRACTIONAL, THEN WILL $(a+b)^m$ REPRESENT ANY BINOMIAL, AFFECTED WITH ANY EXPONENT, AND

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{1 \cdot 2}a^{m-2}b^2$$
$$+ \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}a^{m-3}b^3$$
$$+ \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4}a^{m-4}b^4$$
$$+ \frac{m(m-1)(m-2)(m-3)(m-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}a^{m-5}b^5, + \text{etc.}$$

This is the celebrated BINOMIAL FORMULA, or THEOREM. Its demonstration will be found in the subsequent part of the work. At this stage of his progress the student should learn the formula and become expert in applying it.

185. COR. 1.—*The expansion of a binomial terminates only when the exponent is a positive integer, since only when* m *is a positive integer will a factor of the form* m(m – 1) (m – 2) (m – 3), *etc., become* 0, *as is evident by inspection.*

186. COR. 2.—*When* m *is a positive integer, that is when a binomial is raised to any power, there is one more term in the development than units in the exponent.*

Since the first coefficient is 1; the 2d, m; the 3d, $\frac{m(m-1)}{2}$; the 4th, $\frac{m(m-1)(m-2)}{2 \cdot 3}$; the 5th, $\frac{m(m-1)(m-2)(m-3)}{2 \cdot 3 \cdot 4}$; etc., we notice that the last factor is $m -$ (the number of the term $- 2$); and the number of the term, therefore, which has $m - m$ as a factor is the $(m+2)$th term. But this is 0. Hence the $(m+1)$th term is the last.

187. COR. 3.—*When* m *is a positive integer, the coefficients equally distant from the extremes are equal.*

Thus $(a+b)^m = (b+a)^m$; the former of which gives $a^m + ma^{m-1}b + \frac{m(m-1)}{2}a^{m-2}b^2 +$, etc., and the latter $b^m + mb^{m-1}a + \frac{m(m-1)}{2}b^{m-2}a^2 +$, etc. Whence it appears that the first half of the terms and the last half are exactly symmetrical.

188. COR. 4.—*The sum of the exponents in each term is the same as the exponent of the power.*

SCH.—The last two corollaries apply to the form $(x+y)^m$, and not to such forms as $(2a^3 - 3b^2)^m$, after the latter is fully expanded.

189. Cor. 5.—*A convenient rule for writing out the powers of binomials may be thus stated:*

1st. *There is one more term in the development than there are units in the exponent of the power.*

2d. *The* FIRST *contains only the first letter of the binomial, and the last term only the second, while all the other terms contain both the letters.*

3d. *The exponent of the first letter of the binomial in the first term of the development is the same as the exponent of the required power and* DIMINISHES *by unity to the right, while the exponent of the second letter begins at unity in the second term of the expansion and* INCREASES *by unity to the right, becoming, in the last term, the same as the exponent of the power.*

4th. *The coefficient of the first term of the expansion is unity; of the second, the exponent of the required power; and that of any other term may be found by multiplying the coefficient of the preceding term by the exponent of the first letter in that term, and dividing the product by the exponent of the second letter* + 1.

190. Cor. 6.—*If the sign between the terms of the binomial is minus, as* $(a - b)^m$, *the odd terms of the expansion are* + *and the even ones* —. *This arises from the fact that the odd terms involve even powers of the second or negative term of the binomial, and the even terms involve the odd powers of the same.*

Examples.

1. What is the square of $3a^3$? Of $-2a^{\frac{1}{5}}x$? Of $\frac{3}{5}x^{-\frac{1}{4}}$? Of $-\frac{3}{7}a^{\frac{1}{2}}x$? Of $2\frac{1}{2}\sqrt{x}$? Of $\frac{1}{2}\sqrt{2}$? Of $-\frac{m^2}{x^{\frac{1}{5}}}$?

2. What is the square of $1 - x + x^2$? Of $2a - 3x^3$?

3. Expand the following: $(3-2x-x^2)^2$, $(3x^2-1)^3$, $(x-y+z)^3$, $(1-x^{\frac{1}{2}})^2$, $(x^{\frac{1}{2}}-y^{\frac{1}{2}})^3$.

4. Affect $3a^{\frac{1}{2}}x^{\frac{1}{6}}$ with the exponent 4; $4a^2x^6$ with the exponent 2; a^3x with the exponent $-m$, with the exponent $\frac{1}{2}$, $\frac{2}{3}$; $5x^{\frac{1}{2}}y$ with the exponent $\frac{2}{3}$, $\frac{m}{n}$, -3.

5. Perform the following operations and explain each as a process of factoring, according to (DEM. ***183***): $(125a^{\frac{3}{2}}x^3)^{\frac{2}{3}}$, $(64a^6x)^{\frac{3}{2}}$, $(10a^{-\frac{1}{2}}y)^{\frac{3}{5}}$, $(41m^{-n}y^2)^{-\frac{2}{3}}$, $(a^{\frac{1}{3}}x^{\frac{2}{5}}y^{-\frac{1}{2}})^{-\frac{3}{7}}$, $(a^m b^{\frac{m}{n}} c^{-\frac{s}{r}})^{-\frac{p}{q}}$.

6. Expand the following by the Binomial Formula: $(x+y)^7$, $(x-y)^4$, $(3a^2-x)^3$, $(x+y)^{-5}$, $(x-y)^{-4}$, $(5+x^2)^{\frac{1}{2}}$, $(x^2-a^2)^{-\frac{1}{2}}$, $(1-x^2)^4$, $\sqrt{a^2-a^2e^2}$, $\dfrac{1}{(m-x)^3}$, $\dfrac{1}{\sqrt{1-x^2}}$, $(a^2+bx^2)^{\frac{1}{2}}$.

Three results.

$$\sqrt{a^2-a^2e^2} = a\sqrt{1-e^2} = a(1 - \tfrac{1}{2}e^2 - \frac{1}{2\cdot 4}e^4 - \frac{1\cdot 3}{2\cdot 4\cdot 6}e^6 - \text{etc.})$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{5}{16}x^6 + \tfrac{35}{128}x^8 +, \text{etc.}$$

$$(a^2+bx^2)^{\frac{1}{2}} = a + \frac{bx^2}{2a} - \frac{b^2x^4}{8a^3} + \frac{b^3x^6}{16a^5} -, \text{etc.}$$

7. Write out by COR'S. 5 and 6, the expansions of the following: $(a+b)^5$, $(a-b)^7$, $(a^2-b^2)^3$, $(x^{\frac{1}{2}}-y^{\frac{1}{3}})^4$, $(a^2+y^2)^5$, $(x^{-\frac{1}{2}}-y^{-1})^4$.

SECTION II.

EVOLUTION.

191. Prob. 1.—*To extract any root of a perfect power of that degree.*

RULE.—RESOLVE THE NUMBER INTO ITS PRIME FACTORS, AND SEPARATE THESE INTO AS MANY EQUAL GROUPS AS THERE ARE UNITS IN THE DEGREE OF THE ROOT REQUIRED; THE PRODUCT OF ONE OF THESE GROUPS IS THE ROOT SOUGHT.

192. SCH.—The sign of an even root of a positive number is ambiguous (that is + or −), since an even number of factors gives the same product whether they are positive or negative (***79, 80***). The sign of an odd root is the same as that of the number itself, since an odd number of positive factors gives a positive product and an odd number of negative factors gives a negative product (***80, 81***).

193. COR. 1.—*The roots of monomials can be extracted by extracting the required root of the coefficient and dividing the exponent of each letter by the index of the root, since to extract the square* *root is to affect a number with the exponent $\frac{1}{2}$, the cube root $\frac{1}{3}$, the* n*th root* $\frac{1}{n}$, *etc.* (***183***).

194. Cor. 2.—*The root of the product of several numbers is the same as the product of the roots.*

Thus, $\sqrt[m]{abcx} = \sqrt[m]{a} \cdot \sqrt[m]{b} \cdot \sqrt[m]{c} \cdot \sqrt[m]{x}$, since to extract the mth root of $abcx$ we have but to divide the exponent of each letter by m, which gives, $a^{\frac{1}{m}}b^{\frac{1}{m}}c^{\frac{1}{m}}x^{\frac{1}{m}}$, or $\sqrt[m]{a} \cdot \sqrt[m]{b} \cdot \sqrt[m]{c} \cdot \sqrt[m]{x}$.

195. Cor. 3.—*The root of the quotient of two numbers is the same as the quotient of the roots.*

Thus, $\sqrt[r]{\frac{m}{n}}$ is the same as $\frac{\sqrt[r]{m}}{\sqrt[r]{n}}$, since to extract the rth root of $\frac{m}{n}$ we have but to extract the rth root of numerator and denominator, which operation is performed by dividing their exponents by r. Hence $\sqrt[r]{\frac{m}{n}} = \frac{m^{\frac{1}{r}}}{n^{\frac{1}{r}}} = \frac{\sqrt[r]{m}}{\sqrt[r]{n}}$.

Examples.

1. Extract the square root of each of the following numbers by resolving them into their factors, *i. e.* by (***191***): 222784; 21316; and 5499025.

2. Extract the square root of each of the following, as above: $81a^4x^{-2}y^{\frac{2}{3}}z^{-\frac{1}{5}}$, $a^4c^2+2a^3bc^2+a^2b^2c^2$, $m^4-2m^4x+m^4x^2$.

3. Extract as above: $\sqrt{25a^4b^3}$, $\sqrt{64a^{-6}x^{\frac{2}{3}}}$, $\sqrt{49xy^{\frac{1}{3}}}$, $\sqrt{144a^4m^6}$, $\sqrt{\frac{49a^4b^2}{36m^2n}}$, $\sqrt[4]{16n^{-\frac{1}{2}}y^8}$, $\sqrt[3]{125m^6x^{12}}$, $\sqrt[3]{1728x^2y^{\frac{3}{5}}}$, $\sqrt[5]{-32a^{10}y^{-5}}$.

4. Solve exercises 2 and 3 also by (***193***).

5. Show as in (***194***) that $\sqrt[3]{8 \times 27} = \sqrt[3]{8} \times \sqrt[3]{27}$; also that $\sqrt[5]{a^{-m}b^{\frac{1}{n}}} = \sqrt[5]{a^{-m}} \times \sqrt[5]{b^{\frac{1}{n}}}$.

6. Is $\sqrt{a \pm b} = \sqrt{a} \pm \sqrt{b}$? Is $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$? Is $\sqrt{ab} = \sqrt{a}\sqrt{b}$? Why does the reasoning in the cases which are true not apply to the others? State the true propositions; also the false assumption.

Sch.—The extraction of roots by resolving numbers into their factors according to this rule, is limited in its application for several reasons. In the case of decimal numbers we can always find the prime factors by trial, and hence if the number is an exact power, can get its root. But in case the number is not an exact power of the degree required, we have no method of approximating to its exact root by this rule, as we have by the common method already learned in arithmetic. In case of literal numbers the difficulty of detecting the polynomial factors of a polynomial is usually insuperable. Hence we seek general rules which will not be subject to these objections.

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196. Prob. 2.—*To extract roots whose indices are composed of the factors* 2 *and* 3.

SOLUTION.—To extract the 4th root, extract the square root of the square root. Since the 4th root is one of the 4 equal factors into which a number is conceived to be resolved, if we first resolve a number into 2 equal factors (that is, extract the square root) and then resolve one of these factors into 2 equal factors (that is, extract its square root) one of the last factors is one of the 4 equal factors which compose the original number, and hence the 4th root. In like manner the 6th root is the cube root of the square root, etc.

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197. Prob. 3.—*To extract the* m*th* (*any*) *root of a number.*

SOLUTION.—Instead of giving in detail the demonstrations of the processes for the extraction of roots, we assume that the student is familiar with the subject as presented in common arithmetic,* and propose here to show him how to see a rule for the extraction of *any* root of a decimal number, and of a polynomial, in the expansion of a binomial. Thus

For the		
Square root	$(a+b)^2=a^2+(2a+b)b$	gives the rule;
Cube "	$(a+b)^3=a^3+(3a^2+3ab+b^2)b$	" " "
Fourth "	$(a+b)^4=a^4+(4a^3+6a^2b+4ab^2+b^3)b$	" " "
Fifth "	$(a+b)^5=a^5+(5a^4+10a^3b+10a^2b^2+5ab^3+b^4)b$	" " "
	etc., etc., etc.	

In all cases a represents the part of the root already found, and b the next figure or term of the root; observing that in decimal numbers, a is *tens* with reference to b.

The method of pointing off decimal numbers into periods, and the reason, are shown for the square and cube root in common arithmetic; and the same reasoning extends to other roots.

A polynomial must be arranged as for division, since this is the form which a power takes when the root is similarly arranged.

The solution of a few examples will familiarize the student with this method.

* The whole subject is fully presented in the COMPLETE SCHOOL ALGEBRA.

EXAMPLES.

1. Extract the square root of 7284601.

SOLUTION.

The formula is $(a + b)^2 = a^2 + (2a + b)b$.

		7284601	2699
At first a^2 = the greatest square in 7. $\therefore a = 2$.		4
$2a = 2(20)$ = the *Trial* Divisor	40	328
$\therefore 328 \div 40 = 8$ is the *probable** second root figure.		
$(2a + b) = 40 + 8$ is the True Divisor if 8 is the second root figure. But $48 \times 8 = 384$. $\therefore$ 8 is too large. We will try 6 as the second root figure	6	
Whence $(2a + b)$ = the *True* Divisor	46	276
Now, $2a = 2(260)$ = the *Trial* Divisor	520	5246
$\therefore 5246 \div 520$ = the *probable* next root figure	9	
$(2a + b) = 520 + 9$ = the *True* Divisor	529	4761
Again, $2a = 2(2690)$ = the *Trial* Divisor	5380	48501
$\therefore 48501 \div 5380$ = the probable next root figure	9	
$(2a + b) = 5380 + 9$ = the *True* Divisor	5389	48501

2. Extract the cube root of 99252847.

SOLUTION.

The formula is $(a + b)^3 = a^3 + (3a^2 + 3ab + b^2)b$.

		99252847	463
At first a^3 = the greatest cube in 99. $\therefore a = 4$.		64
$3a^2 = 3(40)^2$ = the *Trial* Divisor	4800	35252
$\therefore 35252 \div 4800 = 7$, the *probable* next root figure.		
$(3a^2 + 3ab + b^2) = 4800 + 840 + 49 = 5689$, the *True* Divisor if 7 is the next root figure. But, as this does not go 7 times in 35252, 7 is too large; and we try 6.		
Now, the corrections to be added to the trial divisor to make the true divisor, are $3ab = 3(40)6 =$	720	
and $b^2 = (6)^2 =$	36	
Hence the true divisor is	5556	33336
New *Trial* Divisor, $3a^2 = 3(460)^2 =$	634800	1916847
Corrections: $3ab = 3(460)3 =$	4140	
$b^2 = (3)^2 =$	9	
True Divisor	638949	1916847

* The new root figure cannot be larger than this quotient. It is often not so large, and the probability of its being considerably less increases with the degree of the root we are extracting.

3. Extract the 5th root of 36936242722357.

SOLUTION.

Formula: $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
$= a^5 + [5a^4 + 10a^3b + 10a^2b^2 + 5ab^3 + b^4]b.$

At first a^5 = the greatest 5th power in 3693. $\therefore a = 5$.			36936242722357 | 517
			3125
Trial Divisor:	$5a^4 = 5(50)^4 = \ldots$	31250000	56862427
Corrections:	1st. $10a^3b = 10(50)^3 \times 1 = \ldots$	1250000	
	2d. $10a^2b^2 = 10(50)^2 \times 1^2 = \ldots$	25000	
	3d. $5ab^3 = 5(50) \times 1^3 = \ldots$	250	
	4th. $b^4 = 1^4 = \ldots$	1	
True Divisor:	$\ldots$	32525251	32525251
Trial Divisor:	$5a^4 = 5(510)^4 = \ldots$	338260050000	2433717622357
Corrections:	1st. $10a^3b = 10(510)^3 \times 7 = \ldots$	9285570000	
	2d. $10a^2b^2 = 10(510)^2 \times 7^2 = .$	127449000	
	3d. $5ab^3 = 5(510) \times 7^3 = \ldots$	874650	
	4th. $b^4 = 7^4 = \ldots$	2401	
True Divisor:	$\ldots$	347673946051	2433717622357

4. What is the 7th root of 12311715481324093344?

SOLUTION.

Formula: $(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$
$= a^7 + [7a^6 + 21a^5b + 35a^4b^2 + 35a^3b^3 + 21a^2b^4 + 7ab^5 + b^6]b.$

			12311715481324093344 | 384
			2187
Trial Divisor:	$7a^6 = 7(30)^6 = \ldots$	5103000000	101247154813
Corrections:	1st. $21a^5b = 21(30)^5 \times 8 = \ldots$	4082400000	
	2d. $35a^4b^2 = 35(30)^4 \times 8^2 = \ldots$	1814400000	
	3d. $35a^3b^3 = 35(30)^3 \times 8^3 = \ldots$	483840000	
	4th. $21a^2b^4 = 21(30)^2 \times 8^4 = \ldots$	77414400	
	5th. $7ab^5 = 7(30) \times 8^5 = \ldots$	6881280	
	6th. $b^6 = 8^6 = \ldots$	262144	
		11568197824	92545582592
Trial Divisor:	$7a^6 = 7(380)^6 = \ldots$	21076554688000000	870157222124093344
Corrections:	1st. $21a^5b = 21(380)^5 \times 4 = \ldots$	665575411200000	
	2d. $35a^4b^2 = 35(380)^4 \times 4^2 = .$	11676761600000	
	3d. $35a^3b^3 = 35(380)^3 \times 4^3 = .$	122913280000	
	4th. $21a^2b^4 = 21(380)^2 \times 4^4 = .$	776294400	
	5th. $7ab^5 = 7(380) \times 4^5 = \ldots$	2723840	
	6th. $b^6 = 4^6 = \ldots$	4096	
		21753930553102336	870157222124093344

5. Extract the square root of each of the following numbers: 7225, 9801, 553536, 5764801, 345642, 2, .5, 3, 50, 1.25, 1.6.

6. Extract the cube root of each of the following numbers: 74088, 122097755681, 2936.493568, 61234, 12.5, .64, .08, 2, 5.

7. Extract the 4th root of 52764813. (See ***196.***)

8. Extract the 6th root of 2985984. (See ***196.***)

9. Extract the 8th root of 1679616. (See ***196.***)

10. Extract the 5th root of 5. $\sqrt[5]{5} = 1.37974 -$.

11. Extract the 7th root of 2. $\sqrt[7]{2} = 1.104 +$.

12. Extract the square root of $49x^2y^2 - 30x^3y + 16y^4 - 24xy^3 + 25x^4$.

SOLUTION.

Formula: $(a+b)^2 = a^2 + (2a+b)b$.

$$
\begin{array}{l|l|l}
a^2 = 25x^4 & 25x^4 - 30x^3y + 49x^2y^2 - 24xy^3 + 16y^4 & 5x^2 - 3xy + 4y^2 \\
\therefore\ a = 5x^2 & 25x^4 & \\
\hline
2a = \textit{Trial}\ \text{Div.} = 10x^2 & -30x^3y + 49x^2y^2 & \\
b = -30x^3y \div 10x^2 = -3xy & & \\
\therefore\ \textit{True}\ \text{Div.} = 10x^2 - 3xy & -30x^3y + 9x^2y^2 & \\
\hline
2a = \textit{Trial}\ \text{Div.} = 10x^2 - 6xy & 40x^2y^2 - 24xy^3 + 16y^4 & \\
\therefore\ b = 40x^2y^2 \div 10x^2 = 4y^2 & & \\
\text{and } \textit{True}\ \text{Div.} = 10x^2 - 6xy + 4y^2 & 40x^2y^2 - 24xy^3 + 16y^4 & \\
\hline
\end{array}
$$

CONDENSED SOLUTION.

$$
\begin{array}{r|l|l}
 & 25x^4 - 30x^3y + 49x^2y^2 - 24xy^3 + 16y^4 & 5x^2 - 3xy + 4y^2 \\
 & 25x^4 & \\
\hline
10x^2 - 3xy & -30x^3y + 49x^2y^2 & \\
 & -30x^3y + 9x^2y^2 & \\
\hline
10x^2 - 6xy + 4y^2 & 40x^2y^2 - 24xy^3 + 16y^4 & \\
 & 40x^2y^2 - 24xy^3 + 16y^4 & \\
\hline
\end{array}
$$

13. Extract the cube root of $36a^4c + 27a^4bc^2x - 8a^3 + b^3x^3 - 6ab^2x^2 + 27a^6c^3 - 36a^3bcx - 54a^5c^2 + 9a^2b^2cx^2 + 12a^2bx$.

SOLUTION.

Formula: $(m+n)^3 = m^3 + (3m^2 + 3mn + n^2)n.$*

$$m^3 = 27a^6c^3 \qquad 27a^6c^3 - 54a^5c^2 + 36a^4c - 8a^3 + 27a^4bc^2x - 36a^3bcx + 12a^2bx + 9a^2b^2cx^2 - 6ab^2x^2 + b^3x^3 \;\big|\; 3a^2c - 2a + bx$$

$$\therefore\; m = 3a^2c \qquad 27a^6c^3$$

Trial Divisor:

$$3m^2 = 3 \times (3a^2c)^2 = 27a^4c^2 \qquad -54a^5c^2 + 36a^4c - 8a^3$$

Corrections:

$$3mn = 3(3a^2c)(-2a) = \quad -18a^3c$$

$$n^2 = (-2a)^2 = \quad 4a^2$$

True Divisor:

$$27a^4c^2 - 18a^3c + 4a^2 \qquad -54a^5c^2 + 36a^4c - 8a^3$$

Trial Divisor:

$$3m^2 = 3(3a^2c - 2a)^2 = 27a^4c^2 - 36a^3c + 12a^2 \qquad 27a^4bc^2x - 36a^3bcx + 12a^2bx + 9a^2b^2cx^2 - 6ab^2x^2 + b^3x^3$$

Corrections:

$$3mn = 3(3a^2c - 2a)bx = \quad 9a^2bcx - 6abx$$

$$n^2 = (bx)^2 = \quad b^2x^2$$

True Divisor:

$$27a^4c^2 - 36a^3c + 12a^2 + 9a^2bcx - 6abx + b^2x^2 \qquad 27a^4bc^2x - 36a^3bcx + 12a^2bx + 9a^2b^2cx^2 - 6ab^2x^2 + b^3x^3$$

14. Extract the square root of each of the following: $8x + 4 + x^4 + 4x^3 + 8x^2$, $25a^2x^2 - 12ax^3 + 16a^4 + 4x^4 - 24a^3x$, $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6$, $x^4 - x^3 + \frac{x^2}{4} + 4x - 2 + \frac{4}{x^2}$, $9x - 24x^{\frac{1}{2}}y^{\frac{2}{3}} + 12x^{\frac{1}{2}} + 16y^{\frac{4}{3}} - 16y^{\frac{2}{3}} + 4$, $9a^{-2} - 12a^{-1}b^2 - 6a + 4b^4 + 4a^2b^2 + a^4$.

* Using m and n instead of a and b, to avoid confusion.

15. Extract the cube root of each of the following: $a^3 - 8b^3 + 12ab^2 - 6a^2b$, $5x^3 - 1 - 3x^5 + x^6 - 3x$, $66x^4 + 1 - 63x^3 - 9x + 8x^6 - 36x^5 + 33x^2$, $60c^2x^4 + 48cx^5 - 27c^6 + 108c^5x - 90c^4x^2 + 8x^6 - 80c^3x^3$, $204c^4x^2 - 144c^5x + 8x^6 - 36cx^5 - 171c^3x^3 + 64c^6 + 102c^2x^4$, $27x - 8x^{\frac{3}{2}} - 36 + 36x^{\frac{5}{2}} + 12x^{-1} - 54x^{\frac{7}{2}} + 9x^{-\frac{5}{2}} + 27x^{\frac{9}{2}} + x^{-6} - 6x^{-\frac{7}{2}}$.

16. What is the 4th root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$?

17. What is the 6th root of $729 - 2916x^2 + 4860x^4 - 4320x^6 + 2160x^8 - 576x^{10} + 64x^{12}$?

[NOTE.—Solve the 16th and 17th both by (***197***) and (***196***)].

18. Find the fifth root of $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$; also of $x^{-20} + 15x^{-16} - 5x^{-14} + 90x^{-12} - 60x^{-10} + 280x^{-8} - 270x^{-6} + 495x^{-4} - 550x^{-2} + 513 - 465x^2 + 275x^4 - 90x^6 + 15x^8 - x^{10}$.

19. Find the 6th root of $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$ by (***196***).

SECTION III.

CALCULUS OF RADICALS.

REDUCTION.

198. Prob. 1.—*To simplify a radical by removing a factor.*

RULE.—RESOLVE THE NUMBER UNDER THE RADICAL SIGN INTO TWO FACTORS, ONE OF WHICH SHALL BE A PERFECT POWER OF THE DEGREE OF THE RADICAL. EXTRACT THE REQUIRED ROOT OF THIS FACTOR AND PLACE IT BEFORE THE RADICAL SIGN AS A COEFFICIENT TO THE OTHER FACTOR UNDER THE SIGN.

DEM.—This process is simply an application of COR., ART. ***194***.

199. COR.—*The denominator of a surd fraction can always be removed from under a radical sign by multiplying both terms of the fraction by some factor which will make the denominator a perfect power of the degree required.*

SCH.—A surd fraction is conceived to be in its simplest form when the smallest possible *whole number* is left under the radical sign.

---

200. Prob. 2.—*To simplify a radical, or reduce it to its lowest terms, when the index is a composite number, and the number under the radical sign is a perfect power of the degree indicated by one of the factors of the index.*

RULE.—EXTRACT THAT ROOT OF THE NUMBER WHICH CORRESPONDS TO ONE OF THE FACTORS OF THE INDEX, AND WRITE THIS ROOT AS A SURD OF THE DEGREE OF THE OTHER FACTOR OF THE GIVEN INDEX.

DEM.—The *mn*th root is one of the *mn* equal factors of a number. If, now, the number is resolved first into *m* equal factors, and then one of these *m* factors is again resolved into *n* other equal factors, one of the latter is the *mn*th root of the number.

---

201. Prob. 3.—*To reduce any number to the form of a radical of a given degree.*

RULE.—RAISE THE NUMBER TO A POWER OF THE SAME DEGREE AS THE RADICAL, AND PLACE THIS POWER UNDER THE RADICAL SIGN WITH THE REQUIRED INDEX, OR INDICATE THE SAME THING BY A FRACTIONAL EXPONENT.

DEM.—That this process does not change the value of the expression is evident, since the number is first involved to a given power, and then the corresponding root of this power is indicated, the latter, or *indicated* operation, being just the reverse of the former.

202. COR.—*To introduce the coefficient of a radical under the radical sign, it is necessary to raise it to a power of the same degree as the radical; for the coefficient being reduced to the same form as the radical by the last rule, we have the product of two like roots, which is equal to the root of the product.*

---

203. Prob. 4.—*To reduce radicals of different degrees to equivalent ones having a common index.*

RULE.—EXPRESS THE NUMBERS BY MEANS OF FRACTIONAL INDICES. REDUCE THE INDICES TO A COMMON DENOMINATOR. PERFORM UPON THE NUMBERS THE OPERATIONS REPRESENTED BY THE

NUMERATORS, AND INDICATE THE OPERATION SIGNIFIED BY THE DENOMINATOR.

DEM.—The only point in this rule needing further demonstration is, that multiplying numerator and denominator of a fractional index by the same number does not change the value of the expression, *i. e.*, that $x^{\frac{a}{b}} = x^{\frac{ma}{mb}}$. Now, $x^{\frac{a}{b}}$ signifies the product of a of the b equal factors into which x is conceived to be resolved. If we now resolve each of these b equal factors into m equal factors, a of them will include ma of the mb equal factors into which x is conceived to be resolved. Hence ma of the mb equal factors of x equals a of the b equal factors.

[The student should notice the *analogy* between this explanation and that usually given in Arithmetic for reducing fractions to equivalent ones having a common denominator. It is not an *identity*.]

---

204. Prob. 5.—*To reduce a fraction having a monomial radical denominator, or a monomial radical factor in its denominator, to a form having a rational denominator.*

RULE.—MULTIPLY BOTH TERMS OF THE FRACTION BY THE RADICAL IN THE DENOMINATOR WITH AN INDEX WHICH ADDED TO THE GIVEN INDEX MAKES IT INTEGRAL.

---

205. Prob. 6.—*To rationalize the denominator of a fraction when it consists of a binomial, one or both of whose terms are radicals of the second degree.*

RULE.—MULTIPLY BOTH TERMS OF THE FRACTION BY THE DENOMINATOR WITH ONE OF ITS SIGNS CHANGED.

DEM.—In the last two cases the student should be able to show, 1st. That the operation does not change the value of the expression; and, 2d. That it produces the required form. [This is the substance of all demonstrations in *Reductions.*]

---

206. Prop. 1.—*A factor may be found which will rationalize any binomial radical.*

DEM.—If the binomial radical is of the form $\sqrt[n]{(a+b)^m}$, or $(a+b)^{\frac{m}{n}}$, the factor is $(a+b)^{\frac{n-m}{n}}$, according to (**204**).

If the binomial is of the form $\sqrt[m]{a^s} + \sqrt[n]{b^r}$, or $a^{\frac{s}{m}} + b^{\frac{r}{n}}$. Let $a^{\frac{1}{m}} = x$, and $b^{\frac{1}{n}} = y$; whence $a^{\frac{s}{m}} = x^s$, and $b^{\frac{r}{n}} = y^r$. Also let p be the least common multiple of m and n, whence x^{sp} and y^{rp} are rational. But $x^{sp} = a^{\frac{sp}{m}}$, and $y^{rp} = b^{\frac{rp}{n}}$. If now we can find a factor which will render $x^s + y^r$, $x^{sp} \pm y^{rp}$, this will be a factor which will render $a^{\frac{s}{m}} + b^{\frac{r}{n}}$, $a^{\frac{sp}{m}} \pm b^{\frac{rp}{n}}$ which is rational. To find the factor which multiplied by $x^s + y^r$ gives $x^{sp} \pm y^{rp}$, we have only to divide the latter by the former. Now $\frac{x^{sp} \pm y^{rp}}{x^s + y^r} = x^{s(p-1)} - x^{s(p-2)} y^r + x^{s(p-3)} y^{2r} - x^{s(p-4)} y^{3r} + \cdots \pm y^{r(p-1)}$ (*A*), the + sign of the last term to be taken when p is odd, and the − sign when it is even (***119***). Therefore $x^{s(p-1)} - x^{s(p-2)}y^r + x^{s(p-3)}y^{2r} - x^{s(p-4)}y^{3r} + \cdots \pm y^{r(p-1)}$, is a factor which will render $\sqrt[m]{a^s} + \sqrt[n]{b^r}$ rational, x^s being understood to be $a^{\frac{s}{m}}$, and $y^r = b^{\frac{r}{n}}$, and p the L. C. M. of m and n.

If the binomial is $\sqrt[m]{a^s} - \sqrt[n]{b^r}$, the factor is found in a similar manner, and is $x^{s(p-1)} + x^{s(p-2)}y^r + x^{s(p-3)}y^{2r} + \cdots + y^{r(p-1)}$.

---

207. Prop. 2.—*A trinomial of the form* $\sqrt{a} + \sqrt{b} + \sqrt{c}$ *may be transformed into an expression with but one radical term by multiplying it by itself with one of the signs changed, as* $\sqrt{a} + \sqrt{b} - \sqrt{c}$. *The product thus arising may then be treated as a binomial radical by considering the sum of the rational terms as one term, and the radical term as the other.*

Thus, $(\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} - \sqrt{c}) = a + b - c + 2\sqrt{ab}$. Again, $[(a + b - c) + 2\sqrt{ab}] \times [(a + b - c) - 2\sqrt{ab}] = a^2 + b^2 + c^2 - 2ab - 2bc - 2ac$, a rational result.

EXAMPLES.

1. Reduce the following to their simplest forms: $\sqrt{108a^2x^3}$, $\sqrt[3]{320a^5x^2y^3}$, $\sqrt{56x^3y}$, $\sqrt[3]{9317m^4x}$, $\sqrt{2873a^3x^5}$, $\sqrt[3]{112999b^{6m}x^{3m-2}}$, $\sqrt{2646a^{\frac{2}{3}}x^5}$, $(7047x^{10}y^2)^{\frac{1}{5}}$, $\sqrt{x^2 - x^3 + x^4}$, $\sqrt{(x^2 - y^2)\ (x-y)}$, $(x^{m-n}y^n)^{\frac{1}{n}}$, $7\sqrt{363xy^3}$, $(a - b)\,[(a^2 - b^2)(a - b)]^{\frac{1}{2}}$, $5\sqrt[3]{704x^5y^2}$, $ab\sqrt[2]{63a^{-3}x^2b^{-2}}$.

2. Reduce the following to their simplest forms (see SCH. ***199***): $\sqrt{\frac{3}{7}}$, $\sqrt[3]{\frac{2}{3}}$, $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{2}{5}}$, $7\sqrt{\frac{1}{7}}$, $\left[\frac{a + b}{a - b}\right]^{\frac{1}{2}}$, $\frac{5}{8}\sqrt{\frac{12}{5}}$, $\frac{3}{11}\sqrt{\frac{4}{13}}$, $4\sqrt[3]{\frac{a}{b^2}}$, $\sqrt[4]{\frac{2}{3}}$, $\sqrt[6]{\frac{1}{2}}$, and $\frac{x^3 - y^3}{x + y}\sqrt{\frac{3x^2 + 6xy + 3y^2}{5(x^2 - y^2)}}$.

3. Reduce the following to their simplest forms (see ***200***):

$$\sqrt[6]{125a^3x^{12}},\ \sqrt[4]{363a^8x^2},\ \sqrt[8]{112x^{12}y^8},\ \sqrt[6]{-1029x^{12}},$$
$$\sqrt[6]{135a^4x - 405a^3x^2 + 405a^2x^3 - 135ax^4}.$$

4. Reduce $5ax^3$ to the form of the square root; also $7xy$; also $\frac{1}{2}$; also $3a - 2$. Reduce $2x^3y^3$ to the form of the 3d root,—to the form of the 5th root. Reduce $\frac{a}{b}$ to the form of the 4th root,—to the form of the cube root. Reduce $\sqrt{\frac{1}{2}}$ to the form of the cube root,—to the form of the 4th root.

5. Introduce under the radical signs the coefficients in the following expressions:

$2\sqrt{\frac{1}{2}}$, $\frac{2}{3}\sqrt[3]{3}$, $\frac{1}{3}\sqrt{3}$, $\frac{1}{5}\sqrt[3]{25}$, $2x\sqrt{x^3}$, $(x+y)\sqrt[3]{x^3 - 3x^2y + 3xy^2 - y^3}$, $(x + y)\sqrt{x - y}$, $\frac{a^2}{5}\sqrt[3]{125x}$, $a^3\left(1 - \frac{b^2}{a^2}\right)^{\frac{3}{2}}$.

6. Reduce to equivalent forms having a common radical index, $\sqrt{2}$ and $\sqrt[3]{3}$; also $\sqrt{3}$, $\sqrt[3]{5}$; also $\sqrt[3]{2x}$, $\sqrt[6]{3x^2}$, $\sqrt{x}$, and $\sqrt[4]{2x^2}$; also $2\sqrt{c}$, $3\sqrt[3]{x}$, and $\frac{1}{3}\sqrt{5}$; also $3\sqrt{5ax}$, $2\sqrt[3]{2xy}$, $\sqrt[6]{10x}$; also $x - y$ and $(x + y)^{\frac{1}{2}}$. Explain each operation upon the principles of factoring as in (***203***).

7. Prove upon the principles of factoring that $\sqrt{2} = \sqrt[6]{8}$; also that $\sqrt[3]{5} = \sqrt[6]{25}$; also that $\sqrt{3} = \sqrt[6]{27}$.

8. Reduce the following to equivalent forms having rational denominators: $\frac{2a\sqrt{5x}}{\sqrt{3x}}$, $\frac{5}{2\sqrt{a^3}}$, $\frac{\sqrt{a}}{\sqrt{b}}$, $\frac{\sqrt{x}}{\sqrt[3]{y}}$, $\frac{\sqrt[m]{x}}{\sqrt[n]{y}}$, $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt[3]{2}}$, $\frac{3}{\sqrt[3]{5}}$, $\frac{\sqrt{2}}{\sqrt{3}}$, $\frac{\sqrt[3]{4}}{\sqrt[3]{7}}$, $\frac{\sqrt[3]{5}}{\sqrt[6]{3}}$.

9 Reduce the following to equivalent forms having rational denominators: $\frac{\sqrt{x^2 + xy + y^2}}{\sqrt{x - y}}$, $\frac{2x}{3\sqrt[3]{3 - x^2}}$, $\frac{x}{x + \sqrt{y}}$, $\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, $\frac{3}{\sqrt{5} + \sqrt{2}}$, $\frac{3}{\sqrt{3} + \sqrt[3]{5}}$, $\frac{2}{\sqrt[3]{5} - \sqrt[3]{4}}$, $\frac{\sqrt{12} - \sqrt{10}}{\sqrt{6} + \sqrt{5}}$, $\frac{3 + 2\sqrt{2}}{\sqrt{5} - \sqrt{3}}$,

$\frac{x}{\sqrt{a^2+x^2}-x}$, $\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}$, $\frac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x-1}-\sqrt{x+1}}$,

$\frac{\sqrt{x^2+x+1}+\sqrt{x^2+x-1}}{\sqrt{x^2+x+1}-\sqrt{x^2+x-1}}$, $\frac{8}{\sqrt{3}+\sqrt{2}+1}$, and $\frac{2}{\sqrt{5}+\sqrt{3}-\sqrt{2}}$.

10. What factor will rationalize $\sqrt[3]{x}-\sqrt[5]{y}$? What $\sqrt[3]{x^2}-\sqrt[4]{y^3}$? What $\sqrt{8}+\sqrt{3}+\sqrt{5}$?

11. By what must numerator and denominator of $\frac{\left(1+\frac{y^2}{x^2}\right)^{\frac{3}{2}}}{x}$ be multiplied to reduce it to the form of a simple fraction? By what $\frac{\left(1+\frac{y^4}{x^4}\right)^{\frac{3}{4}}}{x^2}$?

12. Introduce the coefficients of each of the following into the parentheses: $8(a^2-x^2)^{\frac{3}{2}}$, $a^5(a+a^2x)^{\frac{5}{2}}$, and $x^6(1-x^2)^{\frac{3}{4}}$.

13. Show that $\frac{a+bx-\sqrt{a^2+b^2x^2}}{a-bx+\sqrt{a^2+b^2x^2}}=\frac{\sqrt{a^2+b^2x^2}-a}{bx}$; also that $\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x-a}}=\sqrt{\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}}$.

SECTION IV.

COMBINATIONS OF RADICALS.

ADDITION AND SUBTRACTION.

208. *Prob. 1.*—*To add or subtract radicals.*

RULE.—IF THE RADICALS ARE SIMILAR, THE RULES ALREADY GIVEN (***66, 71***) ARE SUFFICIENT. IF THEY ARE NOT SIMILAR, MAKE THEM SO BY (***198, 203***), AND COMBINE AS BEFORE. IF THEY CANNOT BE MADE SIMILAR, THE COMBINATIONS CAN ONLY BE INDICATED BY CONNECTING THEM WITH THE PROPER SIGNS.

[NOTE.—The student is presumed to be able to give the demonstrations of the problems and propositions in this section, as they are but a recapitulation of what has preceded.]

MULTIPLICATION.

209. Prop. 1.—*The product of the same root of two or more quantities, equals the like root of their product.* (See **194.**)

210. Prop. 2.—*Radicals of the same degree are multiplied by multiplying the quantities under the radical sign and writing the product under the common sign.*

Similar radicals are multiplied by indicating the root by fractional indices, and, for the product, taking the common number with an index equal to the sum of the indices of the factors. (See **82.**)

211. Prob. 2.—*To multiply radicals.*

RULE.—IF THE FACTORS HAVE NOT THE SAME INDEX, REDUCE THEM TO A COMMON INDEX, AND THEN MULTIPLY THE NUMBERS UNDER THE RADICAL SIGN, AND WRITE THE PRODUCT UNDER THE COMMON SIGN.

DIVISION.

212. Prop.—*The quotient of the same root of two quantities equals the like root of their quotient.*

213. Prob. 3.—*To divide radicals.*

RULE.—IF THE RADICALS ARE OF THE SAME DEGREE, DIVIDE THE NUMBER UNDER THE SIGN IN THE DIVIDEND BY THAT UNDER THE SIGN IN THE DIVISOR, AND AFFECT THE QUOTIENT WITH THE COMMON RADICAL SIGN.

IF THE RADICALS ARE OF DIFFERENT DEGREES, REDUCE THEM TO THE SAME DEGREE BEFORE DIVIDING.

INVOLUTION.

214. Prob. 4.—*To raise a radical to any power.*

RULE.—INVOLVE THE COEFFICIENT TO THE REQUIRED POWER, AND ALSO THE QUANTITY UNDER THE RADICAL SIGN, WRITING THE LATTER UNDER THE GIVEN SIGN.

215. COR.—*To raise a radical to a power whose index is the index of the root, is simply to drop the radical sign.*

EVOLUTION.

216. Prob. 5.—*To extract any required root of a monomial radical.*

RULE.—EXTRACT THE REQUIRED ROOT OF THE COEFFICIENT, AND OF THE QUANTITY UNDER THE RADICAL SIGN SEPARATELY, AFFECTING THE LATTER WITH THE GIVEN RADICAL SIGN. REDUCE THE RESULT TO ITS SIMPLEST FORM.

[NOTE.—This problem should not be taken till after Quadratic Equations.]

217. Prob. 6.—*To extract the square root of a binomial, one or both of whose terms are radicals of the second degree.*

SOLUTION.—Such binomials have either the form $a \pm n\sqrt{b}$ or $m\sqrt{a} \pm n\sqrt{b}$. Now observing that $(x \pm y)^2 = x^2 \pm 2xy + y^2$, we see that if we can separate either term of any such binomial surd into two parts, the square root of the product of which shall be $\frac{1}{2}$ the other term, these two parts may be made the first and third terms of a trinomial (corresponding to $x^2 \pm 2xy + y^2$), and the middle term being the second term of the given binomial, the square root will be the sum or difference of the square roots of the parts into which the first term is separated.

[NOTE.—This process requires the solution of a quadratic equation. Thus to extract the square root of $12 - \sqrt{140}$. Letting x and y represent the terms of the binomial root, we have $x^2 + y^2 = 12$, and $2xy = -\sqrt{140}$. Whence $x = \sqrt{5}$ or $\sqrt{7}$, and $y = \sqrt{7}$ or $\sqrt{5}$, and the root is $\sqrt{5} - \sqrt{7}$. The sign between the terms being determined by the sign of the surd in the given binomial. On this account this subject should be reserved until after the student has studied quadratic equations, or the solution effected by inspection. Thus, in this example $\sqrt{140} = 2\sqrt{35}$. Now $\sqrt{35} = \sqrt{5} \times \sqrt{7}$, and since the sum of the squares of these factors is 12, we have $\sqrt{12 - \sqrt{140}} = \sqrt{5} - \sqrt{7}$.]

EXAMPLES.

1. Add $\sqrt{50}$ and $\sqrt{98}$. Add $\sqrt{112ab^2x}$ and $\sqrt{252ab^2x}$. Add $\sqrt[3]{1372a^4x^5}$ and $\sqrt[3]{500a^3x^5}$. Add $\sqrt{x^2y^3}$ and $\sqrt{a^2y^3}$. Add $\sqrt{1183}$ and $-\sqrt{1008}$. Add $5\sqrt{\frac{2}{3}}$ and $2\sqrt{24}$. Add $\sqrt{\frac{3}{5}}$ and $\frac{1}{5}\sqrt{240}$. Add $\sqrt{\frac{3}{4}}$, $\sqrt{\frac{1}{3}}$ and $\frac{1}{2}\sqrt{3}$. Add $\sqrt{18a^5b^2}$, $\sqrt{50a^2b^2}$ and $3\sqrt{18a^3bx^2}$.

2. Show that $x\sqrt{1 + \left(\frac{y}{x}\right)^{\frac{1}{2}}} + y\sqrt{1 + \left(\frac{x}{y}\right)^{\frac{1}{2}}} = (x^{\frac{3}{4}} + y^{\frac{3}{4}})\,(y^{\frac{1}{2}} + x^{\frac{1}{2}})^{\frac{1}{2}}$.

Show that $\sqrt{\dfrac{a^2x - 2ax^2 + x^3}{a^2 + 2ax + x^2}} + \sqrt{\dfrac{a^2x + 2ax^2 + x^3}{a^2 - 2ax + x^2}} = 2\left(\dfrac{a^2 + x^2}{a^2 - x^2}\right)\sqrt{x}$.

Add $\dfrac{a}{a+\sqrt{a^2-x^2}}$ and $\dfrac{a}{a-\sqrt{a^2-x^2}}$; also $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$ and $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$.

3. From $3\sqrt{\frac{2}{5}}$ take $2\sqrt{\frac{1}{10}}$. From $\sqrt[3]{192}$ take $\sqrt[3]{24}$. From $\sqrt{\frac{8}{27}}$ take $\sqrt{\frac{1}{6}}$. From $\frac{3}{4}\sqrt{\frac{2}{3}}$ take $-\frac{2}{5}\sqrt{\frac{1}{6}}$. From $\sqrt[4]{a^6b^2}$ take $\sqrt[3]{x^5y^2}$. From $\sqrt[m]{a^mb}$ take $\sqrt[m]{bx^{2m}}$.

4. Show that $\sqrt[4]{\dfrac{324}{2304}}-\sqrt[6]{\dfrac{8}{421875}}=\frac{11}{60}\sqrt{6}$.

5. Show that $\sqrt{\dfrac{a^2b+2ab^2+b^3}{a^2-2ab+b^2}}-\sqrt{\dfrac{a^2b-2ab^2+b^3}{a^2+2ab+b^2}}=\dfrac{4ab\sqrt{b}}{a^2-b^2}$.

6. Multiply $\sqrt[3]{3}$ by $\sqrt{2}$.* Multiply $\sqrt[3]{2}$ by $\sqrt[5]{3}$. Multiply $\sqrt{\frac{1}{3}}$ by $\sqrt[3]{\frac{3}{5}}$, $\sqrt[4]{2ax}$ by $\sqrt[3]{ax^2}$, $2\sqrt{xy}$ by $3\sqrt[5]{x^3y}$, $\sqrt{1+x^2}$ by $\sqrt[3]{1+x^2}$, $\sqrt[5]{\frac{3}{7}}$ by $\sqrt[3]{\frac{7}{3}}$, $2\sqrt{\frac{1}{2}}$ by $3\sqrt[3]{2}$, $2\sqrt[4]{25}$ by $3\sqrt{5}$, $\sqrt[3]{24}$ by $6\sqrt[6]{3}$.

7. Multiply $9+2\sqrt{10}$ by $9-2\sqrt{10}$, $\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}$ by $\sqrt[3]{x}+\sqrt[3]{y}$, $3\sqrt{5}+2\sqrt{6}-2$ by $2\sqrt{5}+18\sqrt{6}$, $\sqrt[3]{5}-2\sqrt[3]{6}$ by $3\sqrt[3]{4}-\sqrt[3]{36}$.

8. Divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$, $8\sqrt{9}$ by $2\sqrt[3]{3}$, $\sqrt{6}$ by $\sqrt[8]{4}$, $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt[4]{10}$.

9. Divide $2\sqrt{32}+3\sqrt{2}+4$ by $4\sqrt{8}$, $4\sqrt{x^3}$ by $2\sqrt[4]{x}$, $6+2\sqrt{3}-\sqrt[3]{8}$ by $\sqrt{6}$, $\sqrt{ab^2x-b^2cx}$ by $\sqrt{a-c}$, $\sqrt[4]{\dfrac{a}{b}}$ by $\sqrt{\dfrac{a}{b}}$, $(a+b)\sqrt{a^2-1}$ by $(a-b)\sqrt{(a+1)^2}$, $a+b-c+2\sqrt{ab}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$, $\left\{\dfrac{x^2+\sqrt{x^4-a^4}}{x^2-\sqrt{x^4-a^4}}-\dfrac{x^2-\sqrt{x^4-a^4}}{x^2+\sqrt{x^4-a^4}}\right\}$ by $4\sqrt{\dfrac{x^2-a^2}{x^2+a^2}}$.

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* It is of the utmost importance that the pupil be able to give a complete analysis of such examples. Thus, $\sqrt{2}=\sqrt[6]{8}$, since the former is *one* of the *two* equal factors of 2, and the latter is *three* of the *six* equal factors of 2. In like manner $\sqrt[3]{3}=\sqrt[6]{9}$. Consequently $\sqrt{2}\times\sqrt[3]{3}=\sqrt[6]{8}\times\sqrt[6]{9}$. Now since the product of the same root of two numbers is equal to the like root of the product, $\sqrt[6]{8}\times\sqrt[6]{9}=\sqrt[6]{72}$.

10. Raise $3\sqrt[3]{2x^2}$ to the second power. Raise $\frac{1}{3}\sqrt[3]{2ax^2}$ to the 5th power. Cube $-\frac{2}{3}\sqrt{\frac{3}{5}}$. Square $\sqrt{3}-\sqrt{2}$. Cube $3\sqrt{a-x}$. Cube $\sqrt{a}-\sqrt{b}$.

11. Extract the square root of $27\sqrt[3]{135x^6y^4}$; the cube root of $\frac{1}{27}x^6\sqrt{y}$; the 4th root of $25b^4\sqrt{y}$; the 5th root of $224\sqrt[3]{3x^4}$; the cube root of $(1-x)\sqrt{1-x}$; the cube root of $\frac{x}{5}\sqrt{\frac{5}{x}}$; the square root of $\frac{1}{3}\sqrt{\frac{1}{3}}$.

12. Extract the square root of $49+12\sqrt{5}$; of $57+12\sqrt{15}$; of $(a^2+a)x-2ax\sqrt{a}$; $x-2\sqrt{x-1}$; of $\sqrt{18}-4$; of $\frac{a^2}{4}+\frac{c}{2}\sqrt{a^2-c^2}$. (See ***217.***)

SECTION V.

IMAGINARY QUANTITIES.

218. An Imaginary Quantity is an *indicated even* root of a *negative* quantity, or any expression, taken as a whole, which contains such a form either as a factor or a term.

Thus $\sqrt[6]{-x}$, $\sqrt{-y^2}$, $5\sqrt{-x^2}$, $2+\sqrt{-4}$, $\sqrt[4]{-6}$, $3-\sqrt[6]{-1}$, etc., are imaginary quantities.

219. Sch. 1.—It is a mistake to suppose that such expressions are in any proper sense more unreal than other symbols. The term Impossible Quantities should not be applied to them: it conveys a wrong impression. The question is not whether the symbols are symbols of real or unreal (imaginary) quantities or operations, but what interpretation to put upon them, and how to operate with them when they occur.

220. Sch. 2.—A curious property of these symbols, and one which for some time puzzled mathematicians, appears when we attempt to multiply $\sqrt{-x}$ by $\sqrt{-x}$. Now the square root of any quantity multiplied by itself should, by definition, be the quantity itself; hence $\sqrt{-x}\times\sqrt{-x}=-x$. But if we apply the process of multiplying the quantities under the radicals, we have $\sqrt{-x}\times\sqrt{-x}=\sqrt{x^2}=+x$ as well as $-x$. What then is the product of $\sqrt{-x}\times\sqrt{-x}$? Is it $-x$, or is it both $+x$ and $-x$? The true product is $-x$; and the explanation is, that $\sqrt{x^2}$ is, *in general*, $+x$ and $-x$. But when we know what factors were multiplied together to produce x^2, and the nature of our discussion limits us to these, the sign of $\sqrt{x^2}$ is no longer ambiguous: it is the same as was its root.

221. Prop.—*Every imaginary monomial can be reduced to the form* $m\sqrt[2^n]{-1}$, *in which* m *is real* (*not imaginary*). m *may be rational or surd.*

DEM. $x\sqrt[p]{-y}$, p being an even number, is the general symbol for an imaginary monomial. Now if p is a power of 2, we may write at once $p = 2^n$, whence $x\sqrt[p]{-y} = x\sqrt[2^n]{-y} = x\sqrt[2^n]{y(-1)} = x\sqrt[2^n]{y}\ \sqrt[2^n]{-1} = m\sqrt[2^n]{-1}$. If p contains other factors than 2, let r represent their product, and 2^n the product of all the factors of 2 contained in p; whence $p = r2^n$, in which r is odd, since the product of any number of odd factors is odd. We then have $x\sqrt[p]{-y} = x\sqrt[r2^n]{-y} = x\sqrt[r2^n]{y(-1)}$ $= x\sqrt[r2^n]{y}\sqrt[r2^n]{-1} = x\sqrt[r2^n]{y}\ \sqrt[2^n]{\sqrt[r]{-1}} = x\sqrt[r2^n]{y}\ \sqrt[2^n]{-1}$, since any odd root of -1 is -1. Putting $x\sqrt[r2^n]{y} = m$, this becomes $m\sqrt[2^n]{-1}$.

222. SCH.—When $n=1$, *i. e.*, when there is but one factor of 2 in the index of the root, the form becomes $m\sqrt{-1}$. This form is called an imaginary of the *second* degree; $m\sqrt[4]{-1}$ is of the fourth degree, etc. In this discussion we shall confine our attention mainly to imaginaries of the second degree.

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223. Prob.—*To add and subtract imaginary monomials of the second degree, or such as may be reduced to this degree.*

RULE.—REDUCE THEM TO THE FORM $m\sqrt{-1}$, AND THEN COMBINE THEM, CONSIDERING THE SYMBOL $\sqrt{-1}$ AS A SYMBOL OF CHARACTER.*

EXAMPLES.

1. Add $\sqrt{-4}$ and $\sqrt{-9}$.

OPERATION. $\sqrt{-4} = \sqrt{4(-1)} = 2\sqrt{-1}$, and $\sqrt{-9} = 3\sqrt{-1}$.

$$\therefore \sqrt{-4} + \sqrt{-9} = 2\sqrt{-1} + 3\sqrt{-1} = 5\sqrt{-1}.$$

SCH.—The last operation should not be looked upon as taking the sum of 2 times a certain quantity (represented by $\sqrt{-1}$) and 3 times the same quantity, but as 2 *of a certain character* added to 3 of the same character.

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* See (**48**, **49**, **50**). We mean to say that, *as a quantity* (considered numerically), m and $m\sqrt{-1}$, are exactly the same, just as is the case in the expressions $+m$ and $-m$; but that the symbol $\sqrt{-1}$ gives some peculiar or *concrete* significance to m, as does the sign +, or −, or $. *What* this concrete significance is, we cannot here say. It has its clearest interpretation in the Co-ordinate, or General Geometry.

Thus the operation of reducing $\sqrt{-4}$ and $\sqrt{-9}$ to the forms $2\sqrt{-1}$ and $3\sqrt{-1}$ is to be looked upon as rendering the expressions homogeneous. It is, however, to be observed that the symbol $\sqrt{-1}$ is subject, also, to the ordinary laws of number, and may be operated upon accordingly. Thus it has a double significance.

2. Add $4\sqrt{-27}$ and $3\sqrt{-16}$; also $3a\sqrt{-25}$ and $2a\sqrt{-4}$; also $b^3\sqrt{-16}$ and $c\sqrt{-9}$. $b^3\sqrt{-16} + c\sqrt{-9} = (4b^3 + 3c)\sqrt{-1}$.

3. Add $\sqrt{-1024}$ and $\sqrt{-729}$.

OPERATION. $\sqrt{-1024} = \sqrt{1024}\sqrt{-1}$, and $\sqrt{-729} = \sqrt{729}\sqrt{-1}$.
$\therefore \sqrt{-1024} + \sqrt{-729} = 59\sqrt{-1}$.

4. Show that in general $\sqrt[n]{-x} \pm \sqrt[n]{-y} = (\sqrt[n]{x} \pm \sqrt[n]{y})\sqrt[n]{-1}$, when n is any integer.

5. From $\sqrt{-9}$ take $\sqrt{-4}$. From $4\sqrt{-27}$ take $3\sqrt{-16}$. From $3a\sqrt{-25}$ take $2a\sqrt{-4}$. Show that $a\sqrt{-b} - c\sqrt{-d} = (a\sqrt{b} - c\sqrt{d})\sqrt{-1}$.

6. From $\sqrt{-4096}$ take $\sqrt{-9}$. *Rem.* $61\sqrt{-1}$.

SCH.—It would seem improper to omit the 1 before the symbol $\sqrt{-1}$ in such a case as the last, though it has been customary to do so. If we are to consider $\sqrt{-1}$ as a sign of character ("affection," as some say), there is no more reason for omitting the 1 in such a case, than there is in such as $4 - 5 = -1$. That is, if we write $4\sqrt{-1} - 3\sqrt{-1} = \sqrt{-1}$, we ought to write $4 - 5 = -$, or \$5 − \$4 = \$, to be consistent.

7. Show that $\sqrt{-8} + \sqrt{-18} = 5\sqrt{-2} = 5\sqrt{2}\sqrt{-1}$; also that $4\sqrt{-27} + 2\sqrt{-12} = 16\sqrt{3}\sqrt{-1}$; also that $\sqrt[4]{-16} - \sqrt[4]{-1} = 1\sqrt[4]{-1}$; also that $2\sqrt[4]{-a} - \sqrt[4]{-a} = \sqrt[4]{a}\sqrt[4]{-1}$.

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224. Prop.—*Every polynomial containing some real, and some imaginary terms of the second degree, or such as can be reduced to this degree, can be reduced to the form* a ± b$\sqrt{-1}$, *in which* a *and* b *are real.* a *and* b *may be rational or surd.*

DEM.—This is evident from the fact that all the real terms can be combined into one (it may be a polynomial) and represented by a, and the imaginary terms being reduced to the form $m\sqrt{-1}$ can also be combined into one term represented by $\pm b\sqrt{-1}$.

225. Sch.—The form $a \pm b\sqrt{-1}$ is considered the general form of an imaginary quantity of the second degree. When $a = 0$, it becomes the same as $m\sqrt{-1}$. The two expressions $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ are called *Conjugate Imaginaries.* Hence the sum of two conjugate imaginaries is real ($2a$). Also the product of two conjugate imaginaries is real $[(a + b\sqrt{-1}) \times (a - b\sqrt{-1}) = a^2 + b^2$, as will appear hereafter]. The square root of the product of two conjugate imaginaries, taken with the + sign, is called the *Modulus* of each. Thus $+\sqrt{a^2 + b^2}$ is the modulus of both $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$.

Exs.—Find the sum, and the difference of $2 + \sqrt{-1}$ and $3 - \sqrt{-64}$; also of $a + \sqrt{-b}$ and $a + \sqrt{-c}$. *Last results,* $2a + (\sqrt{b} + \sqrt{c})\sqrt{-1}$, $(\sqrt{b} - \sqrt{c})\sqrt{-1}$.

Multiplication and Involution.

226. Prob.—*To determine the character of the product of several imaginary monomial factors of the second degree.*

Solution.—Letting n represent any integer, including 0, $4n$, $4n+1$, $4n+2$, and $4n+3$ may represent all integral exponents, so that all the forms to be treated are $(\sqrt{-1})^{4n}$, $(\sqrt{-1})^{4n+1}$, $(\sqrt{-1})^{4n+2}$, and $(\sqrt{-1})^{4n+3}$. Thus, if $n = 0$, $4n+1 = 1$, $4n+2 = 2$, and $4n+3 = 3$. If $n = 1$, $4n = 4$, $4n+1 = 5$, $4n+2 = 6$, and $4n+3 = 7$. If $n = 2$, $4n = 8$, etc.

We shall now show that
$$(\sqrt{-1})^{4n} = 1,$$
$$(\sqrt{-1})^{4n+1} = \sqrt{-1},$$
$$(\sqrt{-1})^{4n+2} = -1,$$
and
$$(\sqrt{-1})^{4n+3} = -\sqrt{-1}.$$

$$(\sqrt{-1})^{4n} = (\sqrt{-1}\,\sqrt{-1} \times \sqrt{-1}\,\sqrt{-1})^n = [(-1) \times (-1)]^n = 1^n = 1,$$
since $\sqrt{-1}\,\sqrt{-1} = -1$ by (*220*), $(-1)(-1) = 1$, and any power of 1 is 1.

$$(\sqrt{-1})^{4n+1} = (\sqrt{-1})^{4n} \times \sqrt{-1} = 1\sqrt{-1} = \sqrt{-1}, \text{ since } (\sqrt{-1})^{4n} = 1.$$
$$(\sqrt{-1})^{4n+2} = (\sqrt{-1})^{4n} \times (\sqrt{-1})^2 = 1 \times (-1) = -1.$$
$$(\sqrt{-1})^{4n+3} = (\sqrt{-1})^{4n+2} \times \sqrt{-1} = (-1)\sqrt{-1} = -1\sqrt{-1} = -\sqrt{-1}.$$

Ill.—To find what $(\sqrt{-1})^5$ is, we have but to observe that if $n=1$, $4n+1=5$, and $(\sqrt{-1})^5 = (\sqrt{-1})^{4n+1} = \sqrt{-1}$.

Again, $(\sqrt{-1})^3 = (\sqrt{-1})^{4n+3} = -\sqrt{-1}$, since, if $n = 0$, $4n+3 = 3$.

Once more, $(\sqrt{-1})^{10} = (\sqrt{-1})^{4n+2} = -1$, since, if $n = 2$, $4n+2 = 10$.

In like manner, $(\sqrt{-1})^4 = (\sqrt{-1})^{4n} = 1$, since, if $n = 1$, $4n = 4$.

Sch.—In the above we have confined ourselves to the powers of the positive square root, since $-\sqrt{-1}$ may be understood to be $-1\sqrt{-1}$, and the factors -1 be treated separately. Thus, $(-\sqrt{-1})^{4n} = (-1)^{4n}(\sqrt{-1})^{4n} = (\sqrt{-1})^{4n} = 1$. So also $(-\sqrt{-1})^{4n+1} = (-1)^{4n+1} \times (\sqrt{-1})^{4n+1} = (-1)\sqrt{-1} =$

$-\sqrt{-1}$, etc. Or, in other words, giving the $-$ sign to $\sqrt{-1}$ simply changes the signs of the *odd* powers. In examples we shall confine attention to the $+$ root of $\sqrt{-1}$.

EXAMPLES.

1. Multiply $4\sqrt{-3}$ by $2\sqrt{-2}$.

OPERATION. $4\sqrt{-3} = 4\sqrt{3}\sqrt{-1}$, and $2\sqrt{-2} = 2\sqrt{2}\sqrt{-1}$.

$\therefore 4\sqrt{-3} \times 2\sqrt{-2} = 4\sqrt{3}\sqrt{-1} \times 2\sqrt{2}\sqrt{-1} = 8\sqrt{6}(\sqrt{-1})^2 = -8\sqrt{6}.$

2. Show that $\sqrt{-x^2} \times \sqrt{-y^2} = -xy$; also that $3\sqrt{-5} \times 4\sqrt{-3} = -12\sqrt{15}$. What is the product of $-2\sqrt{-2}$ by $-3\sqrt{-3}$?

3. Show that $\sqrt{-2} \times \sqrt{-8} = 4\sqrt{-1}$; also that $\sqrt{-256} \times \sqrt{-27} = 48\sqrt{3}\ \sqrt{-1}$; also that $\sqrt{-2} \times 4\sqrt{-3} = 4\sqrt{6}\ \sqrt{-1}$.

4. Show that $4\sqrt{-1} + \sqrt{-2}$ multiplied by $2\sqrt{-1} - \sqrt{-3}$ equals $\sqrt{6} + 4\sqrt{3} - 2\sqrt{2} - 8$.

5. Show that $\frac{1}{2} - \frac{1}{2}\sqrt{-3}$ squared equals $-\frac{1}{2}(1 + \sqrt{-3})$.

6. Show that $(3-2\sqrt{-4}) \times (5+3\sqrt{-4}) = 39-2\sqrt{-1}$; also that $(1+\sqrt{-1}) \times (1-\sqrt{-1}) = 2$.

7. Show that 2 is the modulus of $\sqrt{2} + \sqrt{-2}$ and $\sqrt{2} - \sqrt{-2}$. What is the modulus of $3+2\sqrt{-3}$ and $3-2\sqrt{-3}$? Of $5-3\sqrt{-1}$ and $5+3\sqrt{-1}$?

8. Show that $(\sqrt{-7})^7 = -7^{\frac{7}{2}}\sqrt{-1}$; also that $(\sqrt{-8})^{16} = 8^8$.

9. What is the 5th power of $2\sqrt{-3}$? Of $3\sqrt{-2}$?

10. What is the product of $\sqrt{-x^2}$, $\sqrt{-y^2}$, $\sqrt{-z^2}$, and $\sqrt{-w^2}$?

DIVISION OF IMAGINARIES.

227. Prob.—*To divide one imaginary of the second degree by another.*

RULE.—REDUCE THE IMAGINARY TERM, OR TERMS, TO THE FORM $m\sqrt{-1}$, OR $m(\sqrt{-1})^n$, AND DIVIDE AS IN DIVISION OF RADICALS, OBSERVING THE PRINCIPLES OF (*226*) TO DETERMINE THE CHARACTER OF THE QUOTIENT OF IMAGINARIES.

EXAMPLES.

1. Divide $\sqrt{-16}$ by $\sqrt{-4}$.

OPERATION. $\sqrt{-16} = 4\sqrt{-1}$, and $\sqrt{-4} = 2\sqrt{-1}$; $\therefore \sqrt{-16} \div \sqrt{-4} = 4\sqrt{-1} \div 2\sqrt{-1} = 2(\sqrt{-1})^0$. But by (*226*) $(\sqrt{-1})^0 = +1$; hence the quotient is 2.

SCH.—A superficial view of the case might make the quotient ± 2. Thus, as the radicals are similar it might be inferred that $\sqrt{-1} \div \sqrt{-1} = \sqrt{\frac{-1}{-1}} = \sqrt{1} = \pm 1$. (See *220*.)

2. Show that $6\sqrt{-3} \div 2\sqrt{-4} = \frac{3}{2}\sqrt{3}$; also that $-\sqrt{-1} \div -6\sqrt{-3} = \frac{1}{18}\sqrt{3}$; also that $1 \div \sqrt{-1} = -\sqrt{-1}$; also that $6 \div 2\sqrt{-1} = -3\sqrt{-1}$.

3. Divide $2\sqrt{-1}$ by $\sqrt[4]{-2}$; also $3\sqrt[6]{-16}$ by -12; also $\sqrt[4]{a}$ by $\sqrt{-1}$.

SUG'S. $2\sqrt{-1} \div \sqrt[4]{-2} = \sqrt[4]{16}\sqrt[4]{(-1)^2} \div \sqrt[4]{2}\sqrt[4]{-1} = \sqrt[4]{8}\sqrt[4]{-1}$.

4. Show that $8\sqrt[6]{-16} \div 2\sqrt[4]{-4} = 4\sqrt[6]{2}\sqrt[4]{-1}$.

5. Show that $(1+\sqrt{-1}) \div (1-\sqrt{-1}) = \sqrt{-1}$; also that $(4+\sqrt{-2}) \div (2-\sqrt{-2}) = 1 + \sqrt{2}\sqrt{-1}$; also that $1 \div (3 - 2\sqrt{-3}) = \frac{3 + 2\sqrt{3}\sqrt{-1}}{21}$; also that $1 \div \frac{a - \sqrt{-x}}{a + \sqrt{-x}} = \frac{a^2 - x + 2a\sqrt{-x}}{a^2 + x}$; also that $\frac{a + \sqrt{-b}}{a - \sqrt{-b}} + \frac{a - \sqrt{-b}}{a + \sqrt{-b}} = \frac{2(a^2 - b)}{a^2 + b}$.

6. Simplify $\frac{(a + b\sqrt{-1})^3 + (a - b\sqrt{-1})^3}{(a + b\sqrt{-1})^2 + (a - b\sqrt{-1})^2}$.

[NOTE.—Here ends the subject of Literal Arithmetic. The student is now prepared for the study of Algebra, properly so-called; *i. e., The Science of the Equation.*]

PART II.

AN ELEMENTARY COURSE IN ALGEBRA.

CHAPTER I.

SIMPLE EQUATIONS.

SECTION I.

EQUATIONS WITH ONE UNKNOWN QUANTITY.

DEFINITIONS.

1. ***An Equation*** is an expression in mathematical symbols, of equality between two numbers or sets of numbers.

2. ***Algebra*** is that branch of Pure Mathematics which treats of the nature and properties of the Equation and of its use as an instrument for conducting mathematical investigations.

3. ***The First Member*** of an equation is the part on the left hand of the sign of equality. ***The Second Member*** is the part on the right.

4. ***A Numerical Equation*** is one in which the *known* quantities are represented by decimal numbers.

5. ***A Literal Equation*** is one in which some or all of the known quantities are represented by letters.

6. ***The Degree*** of an Equation is determined by the highest number of unknown factors occurring in any term, the equation being freed of fractional or negative exponents, as affecting the unknown quantity.

7. ***A Simple Equation*** is an equation of the first degree.

8. ***A Quadratic Equation*** is an equation of the second degree.

9. ***A Cubic Equation*** is an equation of the third degree.

10. Equations above the second degree are called ***Higher Equations.*** Those of the fourth degree are sometimes called ***Biquadratics.***

TRANSFORMATION OF EQUATIONS.

11. To Transform an equation is to change its form without destroying the equality of the members.

12. There are *four* principal transformations of simple equations containing one unknown quantity, viz: *Clearing of Fractions, Transposition, Collecting Terms,* and *Dividing* by the coefficient of the unknown quantity.

13. These transformations are based upon the following

AXIOMS.

AXIOM 1.—*Any operation may be performed upon any term or upon either member, which does not affect the value of that term or member, without destroying the equation.*

AXIOM 2.—*If both members of an equation are increased or diminished alike, the equality is not destroyed.*

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14. Prob.—*To clear an equation of fractions.*

RULE.—MULTIPLY BOTH MEMBERS BY THE LEAST OR LOWEST COMMON MULTIPLE OF ALL THE DENOMINATORS.

DEM.—This process clears the equation of fractions, since, in the process of multiplying any particular fractional term, its denominator is one of the factors of the L. C. M. by which we are multiplying; hence dropping the denominator multiplies by this factor, and then this product (the numerator) is multiplied by the other factor of the L. C. M.

This process does not destroy the equation, since both members are increased or diminished alike.

ILL.—An equation is aptly compared to a pair of scales with equal arms, kept in balance by weights in the two pans.

TRANSPOSITION.

15. Transposing a term is changing it from one member of the equation to the other without destroying the equality of the members.

16. Prob.—*To transpose a term.*

RULE.—DROP IT FROM THE MEMBER IN WHICH IT STANDS AND INSERT IT IN THE OTHER MEMBER WITH THE SIGN CHANGED.

DEM.—If the term to be transposed is +, dropping it from one member diminishes that member by the amount of the term, and writing it with the − sign in the other member, takes its amount from that member; hence both members are diminished alike, and the equality is not destroyed. (Repeat AXIOM 2.)

2d. If the term to be transposed is −, dropping it *increases* the member from which it is dropped, and writing it in the other member with the + sign *increases* that member by the same amount; and hence the equality is preserved. (Repeat AXIOM 2.)

17. To ***Solve*** an equation is to find the value of the unknown quantity; that is, to find what value it must have in order that the equation be true.

18. An equation is said to be ***Satisfied*** for a value of the unknown quantity which makes it a true equation; *i. e.*, which makes its members equal.

19. To ***Verify*** an equation is to substitute the supposed value of the unknown quantity and thus see if it satisfies the equation.

SCH. 2.—The pupil must not understand that the verification is at all necessary to prove that the value found is the correct one. This is demonstrated as we go along, in obtaining it. The object of the verification is to give the pupil a clearer idea of the meaning of an equation, and to detect errors in the work.

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20. Prob. 1.—*To solve a simple equation with one unknown quantity.*

RULE.—1. IF THE EQUATION CONTAINS FRACTIONS, CLEAR IT OF THEM BY ART. 14.

2. TRANSPOSE ALL THE TERMS INVOLVING THE UNKNOWN QUANTITY TO THE FIRST MEMBER, AND THE KNOWN TERMS TO THE SECOND MEMBER BY ART. 16.

3. UNITE ALL THE TERMS CONTAINING THE UNKNOWN QUANTITY INTO ONE BY ADDITION, AND PUT THE SECOND MEMBER INTO ITS SIMPLEST FORM.

4. DIVIDE BOTH MEMBERS BY THE COEFFICIENT OF THE UNKNOWN QUANTITY.

DEM.—The first step, clearing of fractions, does not destroy the equation, since both members are multiplied by the same quantity (AXIOM 2).

The second step does not destroy the equation, since it is adding the same quantity to both members, or subtracting the same quantity from both members (AXIOM 2).

The third step does not destroy the equation, since it does not change the value of the members (AXIOM 1).

The fourth step does not destroy the equation, since it is dividing both mem bers by the same quantity, and thus changes the members alike (AXIOM 2).

Hence, after these several processes, we still have a true equation. But now the first member is simply the unknown quantity, and the second member is all known. Thus we have *what the unknown quantity is equal to; i. e. its value.*

21. SCH. 1.—*It must be fixed in the pupil's mind that he can make but two classes of changes upon an equation:* viz., SUCH AS DO NOT AFFECT THE VALUE OF THE MEMBERS, OR SUCH AS AFFECT BOTH MEMBERS EQUALLY. *Every operation must be seen to conform to these conditions.*

22. COR. 1.—*All the signs of the terms of both members of an equation can be changed from + to −, or* vice versa, *without destroying the equality, since this is equivalent to multiplying or dividing by* − 1.

23. SCH. 2.—It is not always expedient to perform the several transformations *in the same order* as given in the rule. The pupil should bear in mind that the ultimate object is to so transform the equation that the unknown quantity will stand alone in the first member, taking care that, in doing it, nothing is done which will destroy the equality of the members.

24. SCH. 3.—It often happens that an equation which involves the second or even higher powers of the unknown quantity is still, virtually, a simple equation, since these terms destroy each other in the reduction.

SIMPLE EQUATIONS CONTAINING RADICALS.

25. Many equations containing radicals which involve the unknown quantity, though not primarily appearing as simple equations, become so after being freed of such radicals.

26. Prob. 2.—*To free an equation of radicals.*

RULE.—THE COMMON METHOD IS SO TO TRANSPOSE THE TERMS THAT THE RADICAL, IF THERE IS BUT ONE, OR THE MORE COMPLEX RADICAL, IF THERE ARE SEVERAL, SHALL CONSTITUTE ONE MEMBER,

AND THEN INVOLVE EACH MEMBER OF THE EQUATION TO A POWER OF THE SAME DEGREE AS THE RADICAL. IF A RADICAL STILL REMAINS, REPEAT THE PROCESS, BEING CAREFUL TO KEEP THE MEMBERS IN THE MOST CONDENSED FORM AND LOWEST TERMS.

DEM.—That this process frees the equation of the radical which constitutes one of its members is evident from the fact that a radical quantity is involved to a power of the same degree as its indicated root by dropping the root sign.

That the process does not destroy the equality of the members is evident from the fact that the like powers of equal quantities are equal. Both members are increased or decreased alike.

SUMMARY OF PRACTICAL SUGGESTIONS.

27. In attempting to solve a simple equation, always consider,

1. Whether it is best to clear of fractions first.

2. *Look out for compound negative terms.*

3. If the numerators are polynomials and the denominators monomials, it is often better to separate the fractions into parts.

4. It is often expedient, when some of the denominators are monomial or simple, and others polynomial or more complex, to clear of the most simple first, and after each step see that by transposition, uniting terms, etc., the equation is kept in as simple a form as possible.

5. It is sometimes best to transpose and unite some of the terms before clearing of fractions.

6. Be constantly on the lookout for a factor which can be divided out of both members of the equation, or for terms which destroy each other.

7. It sometimes happens that by reducing fractions to mixed numbers the terms will unite or destroy each other, especially when there are several polynomial denominators.

28. WHEN THE EQUATION CONTAINS RADICALS, SPECIALLY CONSIDER,

1. If there is but *one* radical, by causing it to constitute one member and the rational terms the other, the equation can be freed by involving both members to the power denoted by the index of the radical.

2. If there are two radicals and other terms, make the more complex radical constitute one member, alone, before squaring. Such cases usually require two involutions.

3. If there is a radical denominator, and *radicals of a similar form* occur in the numerators or constitute other terms, it may be best to clear of fractions first, either in whole or part.

4. It is sometimes best to rationalize a radical denominator.

Examples for Practice in Solving Simple Equations.

1. Solve and verify the following: (1.) $40-6x-16=120-14x$. (2.) $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=10$. (3.) $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$. (4.) $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}$. (5.) $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$. (6.) $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}$. (7.) $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$. (8.) $\frac{a(b^2+x^2)}{bx}=ac+\frac{ax}{b}$. (9.) $x^2\sqrt{3}=ax+bx+cx$. (10.) $2.04-0.68y-0.02y=0.01$. (11.) $8.4x-7.6=10+2.2x$.

2. Solve (1.) $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax}$. (2.) $4.8x-\frac{.72x-.05}{.5}=1.6x+8.9$. (3.) $\frac{x+px-qx}{p-q}=\frac{mx-n}{m}$. $\left(x=\frac{n(q-p)}{m}\right)$. (4.) $\frac{bx}{2b-a}-\frac{(3bc+ad)x}{2ab(a+b)}-\frac{5ab}{3c-d}=\frac{(3bc-ad)x}{2ab(a-b)}-\frac{5a(2b-a)}{a^2-b^2}$. $\left(x=\frac{5a(2b-a)}{3c-d}\right)$. (5.) $\frac{ax}{a-b}+4b=\frac{cx}{3a+b}$. $\left(x=\frac{8ab^2+4b^3-12a^2b}{3a^2+ab-ac+bc}\right)$. (6.) $\frac{4m(K^2-5x^2)}{8x}=7mp+\frac{5m(g^2-2x)}{4}$. $\left(x=\frac{2K^2}{28p+5g^2}\right)$. (7.) $\frac{a}{bx}+\frac{c}{dx}+\frac{e}{fx}+\frac{g}{hx}=k$. (8.) $\frac{8.5}{2}-\frac{.2}{x}=4\frac{1}{4}-\frac{1-.1x}{x}$. (9.) $\frac{2-3x}{1.5}-\frac{5x}{1.25}-\frac{2x-3}{9}=\frac{x-2}{1.8}+2\frac{7}{9}$. (10.) $5-x\left(3\frac{1}{2}-\frac{2}{x}\right)=\frac{1}{2}x-\frac{3x-(4-5x)}{4}$.

3. In solving the following be careful to observe the suggestions in (**27**): (1.) $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0.$ (2). $\frac{6x-4}{3}-2=\frac{18-4x}{3}+x.$ (3.) $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}.$ (4.) $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$

4. Solve the following, giving special heed to the suggestions in (**28**): (1.) $\sqrt{x-32}=16-\sqrt{x}.$ (2.) $\frac{x}{\sqrt{b^2+x}-b}=c.$ (3.) $\sqrt{x}+\sqrt{x-7}=\frac{21}{\sqrt{x-7}}.$ (4.) $\frac{5x-9}{\sqrt{5x}+3}-1=\frac{\sqrt{5x}-3}{2}.$ (5.) $\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}=\sqrt{x}.$ (6.) $\sqrt{(1+a)^2+(1-a)x}+\sqrt{(1-a)^2+(1+a)x}=2a.$ (7.) $\sqrt{\{13+\sqrt{[7+\sqrt{(3+\sqrt{x})}]}\}}=4.$ (8.) $\sqrt{1+\sqrt{(3+\sqrt{6x})}}=2.$ (9.) $\frac{\sqrt{6x}-2}{\sqrt{6x}+2}=\frac{4\sqrt{6x}-9}{4\sqrt{6x}+6}.$ (10.) $\frac{243+324\sqrt{3x}}{16x-3}=16x-8\sqrt{3x}+3.$ (11.) $\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b.$ (12.) $\frac{ax-1}{\sqrt{ax}+1}=4+\frac{\sqrt{ax}-1}{2}.$

(13.) $\frac{3\sqrt{x}-4}{2+\sqrt{x}}=\frac{15+\sqrt{9x}}{40+\sqrt{x}}.$ (14.) $\frac{\sqrt{ax}+\sqrt{b}}{\sqrt{ax}-\sqrt{b}}=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{b}}.$

(15). $\frac{\sqrt{a}-\sqrt{a-\sqrt{a^2-ax}}}{\sqrt{a}+\sqrt{a-\sqrt{a^2-ax}}}=b.$ (16.) $\frac{\sqrt{m}+\sqrt{m-y}}{\sqrt{m}-\sqrt{m-y}}=\frac{1}{m}.$

(17.) $\frac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b^2$ (18.) $\frac{\sqrt[3]{x+1}-\sqrt[3]{x-1}}{\sqrt[3]{x+1}+\sqrt[3]{x-1}}=\frac{1}{3}.$

(19.) $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b.$ (20.) $\frac{1+x+\sqrt{2x+x^2}}{1+x-\sqrt{2x+x^2}}=a\frac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}.$

(21.) $\frac{\sqrt{3x+1}+\sqrt{3x}}{\sqrt{3x+1}-\sqrt{3x}}=4.$ (22.) $\frac{1}{\sqrt{a-x}+\sqrt{a}}+\frac{1}{\sqrt{a-x}-\sqrt{a}}=\frac{\sqrt{a}}{x}.$

[Several of these equations can be more elegantly reduced by the method given on p. 138, Ex. 47.]

Applications.

29. According to the definition (*2*), Algebra treats of, 1st, The nature and properties of the Equation ; and 2d, the method of using it as an instrument for mathematical investigation.

Having on the preceding pages explained the nature and properties of the equation, we now give a few examples to illustrate its utility as an instrument for mathematical investigation.

30. The Algebraic Solution of a problem consists of two parts:

1st. ***The Statement,*** which consists in expressing by one or more equations the conditions of the problem.

2d. ***The Solution*** of these equations so as to find the values of the unknown quantities in known ones. This process has been explained, in the case of Simple Equations, in the preceding articles.

31. The *Statement* of a problem requires some knowledge of the subject about which the question is asked. Often it requires a great deal of this kind of knowledge in order to "state a problem." This is not Algebra; but it is knowledge which it is more or less important to have according to the nature of the subject.

32. Directions to guide the student in the *Statement of Problems :*

1st. Study the meaning of the problem, so that, *if you had the answer given, you could prove it,* noticing carefully just what operations you would have to perform upon the answer in proving. This is called, *Discovering the relations between the quantities involved.*

2d. Represent the unknown (required) quantities (the answer) by some one or more of the final letters of the alphabet, as x, y, z, or w, and the known quantities by the other letters, or, as given in the problem.

3d. Lastly, by combining the quantities involved, *both known and unknown,* according to the conditions given in the problem (as you would to prove it, if the answer were known) express these relations in the form of an equation.

33. Sch.—It is not always expedient to use x to represent the number sought. The solution is often simplified by letting x be taken for some number from which the one sought is readily found, or by letting $2x$, $3x$, or some multiple of x stand for the unknown quantity. The latter expedient is often used to avoid fractions.

Problems.

1. A's age is double B's, B's is triple C's, and the sum of their ages is 140. Required the age of each.

2. A's age is m times B's, B's is n times C's, and the sum of their ages is s. Required the age of each.

3. The sum of two numbers is 48, and their difference 12. What are the numbers?

4. The sum of two numbers is s, and their difference d. What are the numbers?

5. Having the sum and difference of two numbers given, how do you find the numbers, arithmetically?

6. A post is $\frac{1}{5}$th in the earth, $\frac{3}{7}$ths in the water, and 13 feet in the air. What is the length of the post?

7. A post is $\frac{1}{n}$th in the earth, $\frac{r}{m}$ths in the water, and a feet in the air. What is the length of the post?

8. What fraction is that, whose numerator is less by 3 than its denominator; and if 3 be taken from the numerator, the value of the fraction will be $\frac{1}{4}$?

9. Give the *general* solution of the last; *i. e.*, the solution when the numbers are all represented by letters. Then substitute the above numbers and find the answer to that *special* problem.

Sug.—Letting the numerator be a less than the denominator, and $\frac{m}{n}$ be the fraction after b is taken from the numerator, the fraction is $\frac{am + bn}{an + bn}$.

10. A man sold a horse and chaise for \$200; $\frac{1}{2}$ the price of the horse was equal to $\frac{1}{3}$ the price of the chaise. Required, the price of each. Chaise, \$120; horse, \$80.

Generalize and solve the last, and then by substituting the numbers given in it find the special answers. Treat in like manner the next nine problems.

11. Out of a cask of wine which had leaked away a third part, 21 gallons were afterward drawn, when it was found that one-half remained. How much did the cask hold? *Ans.*, 126 galls.

12. A and B can do a piece of work in 12 days, but when A worked alone he did the same work in 20. How long would it take B to do the same work? *Ans.*, 30 days.

13. A cistern can be filled by 3 pipes; by the first in $1\frac{1}{2}$ hours, by the second in $2\frac{1}{2}$ hours, and by the third in 5 hours. In what time will the cistern be filled, when all are left open at once?

14. Four merchants entered into a speculation, for which they subscribed 4755 dollars; of which B paid three times as much as A; C paid as much as A and B; and D paid as much as C and B. What did each pay?

15. A and B trade with equal stocks. In the first year A tripled his stock and had \$27 to spare; B doubled his stock, and had \$153 to spare. Now the amount of both their gains was five times the stock of either. What was that?

16. A and B began to trade with equal sums of money. In the first year A gained 40 dollars, and B lost 40; but in the second A lost one-third of what he then had, and B gained a sum less by 40 dollars than twice the sum that A had lost; when it appeared that B had twice as much money as A. What money did each begin with? *Ans.*, 320 dollars.

17. What number is that to which if 1, 5, and 13 be severally added, the first sum divided by the second shall equal the second divided by the third?

18. Divide 49 into two such parts that the greater increased by 6 divided by the less diminished by 11, shall be $4\frac{1}{2}$.

19. A cistern which contains 2400 gallons can be filled in 15 minutes by three pipes, the first of which lets in 10 gallons per minute, and the second 4 gallons less than the third. How much passes through each pipe in a minute?

20. Find a number such that, if from the quotient of the number increased by 5, divided by the number increased by 1, we subtract the quotient of 3 diminished by the number, divided by the number diminished by 2, the remainder shall be 2.

21. Divide a into two such parts, that one may be the $\frac{m}{n}$th part of the other.

22. Divide a into two such parts, that the sum of the quotients which are obtained by dividing one part by m, and the other by n, shall be equal to b. *The parts are* $\frac{m(nb-a)}{n-m}$, and $\frac{n(mb-a)}{m-n}$.

23. Letting p represent the principal, i the interest for time t, a the amount, and r the *per cent.* for a unit of time, produce the following *formulæ*, and give their meaning:

(1.) $i = \frac{trp}{100}$;	(3.) $t = \frac{100i}{rp}$;	(5.) $p = \frac{100i}{tr}$;
(2.) $a = p + i = p\frac{100 + tr}{100}$;	(4.) $r = \frac{100i}{tp}$;	(6.) $p = \frac{100a}{100 + tr}$.

24. In what time will a given principal double, triple, or quadruple itself, at 5%? at 6%? at 7%?

25. What is the worth of a note of $500 Nov. 2d, 1872, which is dated Feb. 23d, 1870, bears 12% interest, and is due Jan. 1st, 1875, money being worth 7%? *Ans.*, $687.23 +

26. On a sum of money borrowed, annual interest is paid at 5%. After a time $200 are paid on the principal, and the interest on the remainder is reduced to 4%. By these changes the annual interest is lessened one-third. What was the sum borrowed?

27. An artesian well supplies a manufactory. The consumption of water goes on each week-day from 3 A.M. to 6 P.M. at double the rate at which the water flows into the well. If the well contained 2250 gallons of water when the consumption began on Monday morning, and the well was just emptied at 6 P.M. on the next Thursday evening but one, how many gallons flowed into the well per hour?

28. The hind and fore wheels of a carriage have circumferences 16 and 14 feet respectively. How far has the carriage advanced when the fore wheel has made 51 revolutions more than the other?

29. A merchant gains the first year 15% on his capital; the second year, 20% on the capital at the close of the first; and the third year, 25% on the capital at the close of the second; when he finds that he has cleared $1000.50. Required his capital. *Capital*, $1380.

30. A man had $2550 to invest. He invested part in certain 3% stocks, and part in R. R. shares of $25 each, which pay annual dividends of $1.00 per share. The stocks cost him $81 on a hundred, and the R. R. shares $24 per share; and his income from each source is the same. How many R. R. shares did he buy?

SECTION II.

INDEPENDENT, SIMULTANEOUS, SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES.

DEFINITIONS.

34. *Independent Equations* are such as express *different* conditions, and neither can be reduced to the other.

35. *Simultaneous Equations* are those which express different conditions of the same problem, and consequently the letters representing the unknown quantities signify the same things in each. All the equations of such a group are satisfied by the same values of the unknown quantities.

36. *Elimination* is the process of producing from a given set of simultaneous equations containing two or more unknown quantities, a new set of equations in which one, at least, of the unknown quantities shall not appear. The quantity thus disappearing is said to be eliminated. (The word literally means *putting out of doors*. We use it as meaning *causing to disappear*.)

37. There are ***Five Methods of Elimination,*** viz., by *Comparison*, by *Substitution*, by *Addition or Subtraction*, by *Undetermined Multipliers*, and by *Division.*

ELIMINATION BY COMPARISON.

38. Prob. 1.—*Having given two independent, simultaneous, simple equations between two unknown quantities, to deduce therefrom by Comparison a new equation containing only one of the unknown quantities.*

RULE.—1st. FIND EXPRESSIONS FOR THE VALUE OF THE SAME UNKNOWN QUANTITY FROM EACH EQUATION, IN TERMS OF THE OTHER UNKNOWN QUANTITY AND KNOWN QUANTITIES.

2d. PLACE THESE TWO VALUES EQUAL TO EACH OTHER, AND THE RESULT WILL BE THE EQUATION SOUGHT.

DEM.—The first operations being performed according to the rules for simple equations with one unknown quantity, need no further demonstration.

2d. Having formed expressions for the value of the *same* unknown quantity in both equations, since the equations are simultaneous this unknown quantity means the same thing in the two equations, and hence the two expressions for its value are equal. Q. E. D.

SCH.—The resulting equation can be solved by the rules already given.

ELIMINATION BY SUBSTITUTION.

39. Prob. 2.—*Having given two independent, simultaneous, simple equations, between two unknown quantities, to deduce therefrom by Substitution a single equation with but one of the unknown quantities.*

RULE.—1st. FIND FROM ONE OF THE EQUATIONS THE VALUE OF THE UNKNOWN QUANTITY TO BE ELIMINATED, IN TERMS OF THE OTHER UNKNOWN QUANTITY AND KNOWN QUANTITIES.

2d. SUBSTITUTE THIS VALUE FOR THE SAME UNKNOWN QUANTITY IN THE OTHER EQUATION.

DEM.—The first process consists in the solution of a simple equation, and is demonstrated in the same way.

The second process is self-evident, since, the equations being simultaneous, the letters mean the same thing in both, and it does not destroy the equality of the members to replace any quantity by its equal. Q. E. D.

ELIMINATION BY ADDITION OR SUBTRACTION.

40. Prob. 3.—*Having given two independent, simultaneous, simple equations between two unknown quantities, to deduce therefrom by Addition or Subtraction a single equation with but one unknown quantity.*

RULE.—1st. REDUCE THE EQUATIONS TO THE FORMS $ax + by = m$, AND $cx + dy = n$.

2d. IF THE COEFFICIENTS OF THE QUANTITY TO BE ELIMINATED ARE NOT ALIKE IN BOTH EQUATIONS, MAKE THEM SO BY FINDING THEIR L. C. M. AND THEN MULTIPLYING EACH EQUATION BY THIS L. C. M. EXCLUSIVE OF THE FACTOR WHICH THE TERM TO BE ELIMINATED ALREADY CONTAINS.

3d. IF THE SIGNS OF THE TERMS CONTAINING THE QUANTITY TO BE ELIMINATED ARE ALIKE IN BOTH EQUATIONS, SUBTRACT ONE

EQUATION FROM THE OTHER, MEMBER BY MEMBER. IF THESE SIGNS ARE UNLIKE, ADD THE EQUATIONS.

DEM.—The first operations are performed according to the rules already given for clearing of fractions, transposition, and uniting terms, and hence do not vitiate the equations. The object of this reduction is to make the two subsequent steps practicable.

The second step does not vitiate the equations, since in the case of either equation, both its members are multiplied by the same number.

The third step eliminates the unknown quantity, since, as the terms containing the quantity to be eliminated have the same numerical value, if they have the *same* sign, by *subtracting* the equations one will destroy the other, and if they have different signs, by *adding* the equations they will destroy each other. The result is a true equation, since, If equals (the two members of one equation) are added to equals (the two members of the other equation), the sums are equal. Thus we have a new equation with but one unknown quantity. Q. E. D.

ELIMINATION BY UNDETERMINED MULTIPLIERS.

41. Prob. 4.—*Having given two independent, simultaneous, simple equations between two unknown quantities, to deduce therefrom by Undetermined Multipliers a single equation with but one unknown quantity.*

RULE.—1st. REDUCE THE EQUATIONS TO THE FORMS $ax + by = m$, AND $cx + dy = n$.

2d. MULTIPLY ONE OF THE EQUATIONS BY AN UNDETERMINED FACTOR, AS f, AND FROM THE RESULT SUBTRACT THE OTHER EQUATION, MEMBER BY MEMBER.

3d. IN THE RESULTING EQUATION, PLACE THE COEFFICIENT OF THE UNKNOWN QUANTITY TO BE ELIMINATED EQUAL TO 0; FROM THIS EQUATION FIND THE VALUE OF f, AND SUBSTITUTE IT IN THE OTHER TERMS OF THE EQUATION.

DEM.—[Reason for the first step, same as in the last method.]

Now multiply one of the equations, as $ax + by = m$, by f, and subtract the other, member by member, giving $(af - c)x + (bf - d)y = mf - n$. To eliminate y, put $bf - d = 0$, giving $f = \frac{d}{b}$. This value of f substituted in $(af - c)x + (bf - d)y = mf - n$, will cause the term containing y to disappear by making its coefficient 0, and there will result an equation containing only the unknown quantity x, and known quantities. Q. E. D.

Thus, given $3x+7y=33$, and $2x+4y=26$.

Multiply the 1st by f, - - - - - $3fx+7fy=33f$
Subtract the 2d, - - - - - - - - $2x+\ 4y=20$
And we have - - - - - - - - - $(3f-2)x+(7f-4)y=33f-20$.

Putting $7f-4=0$, $f=\frac{4}{7}$. Substituting, $(3\times\frac{4}{7}-2)x=33\times\frac{4}{7}-20$. Whence, $-\frac{2}{7}x=-\frac{8}{7}$, or $x=4$.

In like manner, putting $3f-2=0$, $f=\frac{2}{3}$. And $(7\times\frac{2}{3}-4)y=33\times\frac{2}{3}-20$. Whence $y=3$.

Elimination by Division.

42. Prob. 5.—*Having given two independent, simultaneous, equations of any degree, between two unknown quantities, to deduce therefrom by Division a single equation with but one unknown quantity.*

RULE.—CLEAR THE EQUATIONS OF FRACTIONS, AND TRANSPOSE ALL THE TERMS TO ONE MEMBER. TREAT THE POLYNOMIALS THUS OBTAINED AS IN THE PROCESS FOR FINDING THE HIGHEST COMMON DIVISOR, CONTINUING THE PROCESS UNTIL ONE UNKNOWN QUANTITY DISAPPEARS FROM THE REMAINDER. PUTTING THIS REMAINDER EQUAL TO 0, WE HAVE THE EQUATION SOUGHT.

DEM.—Since each of the polynomials is equal to 0, any number of times one subtracted from the other (*i. e.* any remainder) is 0.

Examples.

[NOTE.—The pupil should solve the following by each of the preceding methods, so as to make all familiar, and in each instance notice what method is most expeditious.]

(1.) $2x+7y=41$, $3x+4y=42$.

(2.) $x+15y=49$, $3x+7y=71$.

(3.) $6x+4y=236$, $3x+15y=573$.

(4.) $29x-175=14y$, $87x-56y=497$.

(5.) $188-5x-9y=0$, $13x=57+2y$.

(6.) $5x-4=3y$, $10+7x-12y=0$.

(7.) $5y-21=2x$, $13x-4y=120$.

(8.) $7y-3x=139$, $2x+5y=91$.

(9.) $69y-17x=103$, $14x-13y=-41$.

(10.) $\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}$, $\frac{2y+4}{3}=\frac{4x+y+13}{8}$.

(11.) $abx+cdy=2$, $ax-cy=\frac{d-b}{bd}$.

(12.) $\dfrac{a}{b+y}=\dfrac{b}{3a+x}$, $ax+2by=d.$

(13.) $(b+c)(x+c-b)+a(y+a)=2a^2$, $\dfrac{ay}{(b-c)x}=\dfrac{(b+c)^2}{a^2}.$

(14.) $\dfrac{x}{a+b}+\dfrac{y}{a-b}=2a$, and $\dfrac{x-y}{4ab}=1.$

(15.) $2.4x+.32y-\dfrac{.36x-.05}{.5}=.8x+\dfrac{2.6+.005y}{.25}$, $\dfrac{.04y+.1}{.3}=\dfrac{.07x-.1}{.6}.$

(16.) $\dfrac{\frac{2x}{3}-\frac{5y}{12}}{\frac{7}{4}}-\dfrac{\frac{3x}{2}-\frac{y}{3}}{\frac{23}{2}}=2$, and $\dfrac{x-y}{x+y}=\dfrac{1}{5}.$

(17.) $\dfrac{2}{ax}+\dfrac{3}{by}=5$, $\dfrac{5}{ax}-\dfrac{3}{by}=2.$

(18.) $\dfrac{7}{x}+\dfrac{5}{y}=19$, $\dfrac{8}{x}-\dfrac{3}{y}=7.$

(19.) $\dfrac{m}{nx}+\dfrac{n}{my}=m+n$, $\dfrac{n}{x}+\dfrac{m}{y}=m^2+n^2.$

[NOTE.—Solve the following by (***42***).]

20. Eliminate x between the following: $5x+y=105$ and $x+3y=35$; also $\frac{1}{6}(2x+3y)=8-\frac{1}{3}x$ and $11+y=\frac{1}{2}(7y-3x)$; also $\frac{1}{5}(y-2)-\frac{1}{3}(10-y)=\frac{1}{4}(z-10)$ and $\frac{1}{3}(2z+4)-\frac{1}{8}(2y+z)=\frac{1}{4}(y+13)$; also $x^2+6xy=144$ and $6xy+36y^2=432$; also $x^3+y^3=2728$ and $x^2-xy+y^2=124$; also $x^4+x^3y+x^2y^2+xy^3+y^4=1$ and $x^2+y^2=2$; also $x+y+xy=34$ and $x^2+y^2=52$.

SOLUTION OF THE LAST.

$$x+y+xy-34\ )\qquad x^2+y^2-52\,|\,\underline{x+34-y}$$
$$(1+y)x+y-34\ )\ (1+y)x^2+(1+y)y^2-52(1+y)$$
$$\underline{(1+y)x^2+xy-34x}$$
$$\underline{(34-y)x+(y^2-52)(1+y)}$$
$$(1+y)(34-y)x+(y^2-52)(1+y)^2$$
$$\underline{(1+y)(34-y)x-(34-y)^2}$$

Equation sought, $(34-y)^2+(y^2-52)(1+y)^2=0.$

[NOTE.—The equations resulting from the elimination in several of the above cases are of degrees higher than the first, and hence their resolution is not to be expected at this stage of the student's progress.]

APPLICATIONS.

1. A wine merchant has two kinds of wine, one worth 72 cents a quart, and the other 40 cents. How much of each must he put in a mixture of 50 quarts, so that it shall be worth 60 cents a quart?

2. A crew that can pull at the rate of 12 miles an hour down the stream, finds that it takes twice as long to row a given distance up stream as down. What is the rate of the current?

3. A man sculls a certain distance down a stream which runs at a rate of 4 miles an hour, in 1 hour and 40 minutes. In returning it takes him 4 hours and 15 minutes to reach a point 3 miles below his starting place. How far did he scull down the stream, and at what rate could he scull in still water?

4. A man puts out \$10,000 in two investments. For the first he gets 5% and for the second 4%. The first yields annually \$50 more than the second. What is each investment?

[NOTE.—Generalize the statement and solution of the preceding problems.]

5. What fraction is that whose numerator being doubled and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

6. There is a number consisting of two digits, which is equal to four times the sum of those digits; and if 18 be added to it, the digits will be inverted. What is the number?

7. A work is to be printed, so that each page may contain a certain number of lines, and each line a certain number of letters. If we wished each page to contain 3 lines more, and each line 4 letters more, then there would be 224 letters more in each page; but if we wished to have 2 lines less in a page, and 3 letters less in each line, then each page would contain 145 letters less. How many lines are there in each page? and how many letters in each line?

8. A sum of money put out at simple interest amounted to \$5250 in 10 months, and to \$5450 in 18 months. What was the principal, and what the rate?

9. In an alloy of silver and copper, $\frac{1}{m}$ of the whole + p ounces was silver, and $\frac{1}{n}$ of the whole − q ounces was copper. How many ounces were there of each?

10. When a is added to the greater of two numbers, it is m times the less; but when b is added to the less, it is n times the greater. What are the numbers?

11. When 4 is added to the greater of two numbers, it is $3\frac{1}{4}$ times the less; but when 8 is added to the less, it is $\frac{1}{2}$ the greater. What are the numbers? Solve by substituting in the results of the preceding.

12. There is a cistern into which water is admitted by three cocks, two of which are of exactly the same dimensions. When they are all open, five-twelfths of the cistern is filled in 4 hours; and if one of the equal cocks be stopped, seven-ninths of the cistern is filled in 10 hours and 40 minutes. In how many hours would each cock fill the cistern?

13. A banker has two kinds of change; there must be a pieces of the first to make a crown, and b pieces of the second to make the same: now a person wishes to have c pieces for a crown. How many pieces of each kind must the banker give him?

Ans., $\frac{a(b-c)}{b-a}$ of the first kind, $\frac{b(c-a)}{b-a}$ of the second.

14. An ingot of metal which weighs n pounds loses p pounds when weighed in water. This ingot is itself composed of two other metals, which we may call M and M′; now n pounds of M loses q pounds when weighed in water, and n pounds of M′ loses r pounds when weighed in water. How much of each metal does the original ingot contain?

Ans., $\frac{n(r-p)}{r-q}$ pounds of M, $\frac{n(p-q)}{r-q}$ pounds of M′.

SECTION III.

INDEPENDENT, SIMULTANEOUS, SIMPLE EQUATIONS WITH MORE THAN TWO UNKNOWN QUANTITIES.

43. Prob.—*Having given several independent, simultaneous, simple equations, involving as many unknown quantities as there are equations, to find the values of the unknown quantities.*

RULE.—COMBINE THE EQUATIONS TWO AND TWO BY ANY OF THE METHODS OF ELIMINATION, ELIMINATING BY EACH COMBINATION THE SAME UNKNOWN QUANTITY, THUS PRODUCING A NEW SET OF EQUATIONS, ONE LESS IN NUMBER, AND CONTAINING AT LEAST ONE LESS UNKNOWN QUANTITY. COMBINE THIS NEW SET TWO AND TWO IN LIKE MANNER, ELIMINATING ANOTHER OF THE UNKNOWN QUANTITIES. REPEAT THE PROCESS UNTIL A SINGLE EQUATION IS FOUND WITH BUT ONE UNKNOWN QUANTITY. SOLVE THIS EQUATION AND THEN SUBSTITUTE THE VALUE OF THIS UNKNOWN QUANTITY IN ONE OF THE NEXT PRECEDING SET OF EQUATIONS, OF WHICH THERE WILL BE BUT TWO, WITH TWO UNKNOWN QUANTITIES, AND THERE WILL RESULT AN EQUATION CONTAINING ONLY ONE, AND THAT ANOTHER OF THE UNKNOWN QUANTITIES, THE VALUE OF WHICH CAN THEREFORE BE FOUND FROM IT. SUBSTITUTE THE TWO VALUES NOW FOUND IN ONE OF THE NEXT PRECEDING SET, AND FIND THE VALUE OF THE REMAINING UNKNOWN QUANTITY IN THIS EQUATION. CONTINUE THIS PROCESS TILL ALL THE UNKNOWN QUANTITIES ARE DETERMINED.

DEM.—1. The combinations of the equations give true equations because they are all made upon the methods of elimination already demonstrated.

2. That the number of equations can always be reduced to one by this process, is evident, since, if we have n equations and combine any one of them with each of the others, there will be $n-1$ new equations. Combining one of these $n-1$ new equations with all the rest there will result $n-2$. Hence $n-1$ such combinations will produce a single equation; and as one unknown quantity, at least, has disappeared from each set, there will be but one left. Q. E. D.

SCH. 1.—If any equation of any set does not contain the quantity we are seeking to eliminate, this equation can be written at once in the next set, and the remaining equations combined.

SCH. 2.—In eliminating any unknown quantity from a particular set of equations, any one of the equations may be combined with each of the others, and the new set thus formed. But some other order may be preferable as giving simpler results.

Sch. 3.—It is sometimes better to find the values of all the unknown quantities in the same way as the first is found, rather than by substitution.

Examples.

1.

$$x+y+z=31,\quad x+y-z=25,\quad x-y-z=9.$$

2.

$$x+y+z=9,\quad x+3y-3z=7,\quad x-4y+8z=8.$$

3.

$$2x+3y+4z=29,\quad 3x+2y+5z=32,\quad 4x+3y+2z=25.$$

4.

$$\tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{4}z=62,\quad \tfrac{1}{3}x+\tfrac{1}{4}y+\tfrac{1}{5}z=47,\quad \tfrac{1}{4}x+\tfrac{1}{5}y+\tfrac{1}{6}z=38.$$

5.

$$\left.\begin{aligned}x+\tfrac{1}{2}y&=100,\\ y+\tfrac{1}{3}z&=100,\\ z+\tfrac{1}{4}x&=100.\end{aligned}\right\}\quad x=64;\ y=72;\ z=84.$$

6.

$$\left.\begin{aligned}\frac{2x+3y}{2}+2z&=8,\\ x+2y-5z&=2,\\ \frac{5x-6y}{3}+z&=2.\end{aligned}\right\}\quad x=3;\ y=2;\ z=1.$$

7.

$$x+\frac{y+z}{2}=85,\quad y+\frac{x+z}{3}=85,\quad z+\frac{x+y}{4}=85.$$

8.

$$ay+bx=c,\quad cx+az=b,\quad bz+cy=a.$$

9.

$$\frac{2}{x}+\frac{1}{y}=\frac{3}{z},\quad \frac{3}{z}-\frac{2}{y}=2,\quad \frac{1}{x}+\frac{1}{z}=\frac{4}{3}.$$

10.

$$y+z=2yz,\quad x+z=3xz,\quad x+y=4xy.$$

11.

$$\frac{a}{x}+\frac{b}{y}=1,\quad \frac{b}{y}+\frac{c}{z}=1,\quad \frac{c}{z}+\frac{a}{x}=1.$$

12.

$$x+y+z=0,\quad (b+c)x+(c+a)y+(a+b)z=0,\quad bcx+cay+abz=1.$$

13. $xyz=a(yz-zx-xy)=b(zx-xy-yz)=c(xy-yz-zx).$

14.	15.	16.
$3u-2y=2,$	$3x-4y+3z+3v-6u=11,$	$u+v+x+y=10,$
$5x-7z=11,$	$3x-5y+2z-4u=11,$	$u+v+x+z=11,$
$2x+3y=39,$	$10y-3z+3u-2v=2,$	$u+v+y+z=12,$
$4y+3z=41.$	$5z+4u+2v-2x=3,$	$u+x+y+z=13,$
	$6u-3v+4x-2y=6.$	$v+x+y+z=14.$

17. $x+y+z=a+b+c, \quad bx+cy+az=cx+ay+bz=a^2+b^2+c^2.$

Applications.

1. Three persons, A, B, and C, were talking of their guineas; says A to B and C, give me half of yours and I shall have 34; says B to A and C, give me a third part of yours and I shall have 34; says C to A and B, give me a fourth part of yours and I shall have 34. How many had each? *Ans.*, A 10, B 22, C 26.

2. For \$8 I can buy 2 lbs. of tea, 10 lbs. of coffee, and 20 lbs. of sugar, or 2 lbs. of tea, 5 lbs. of coffee, and 30 lbs. of sugar, or 3 lbs. of tea, 5 lbs. of coffee, and 10 lbs. of sugar. What are the prices?

3. A person possesses a certain capital which is invested at a certain rate per cent. A second person has £1000 more capital than the first and invests it at one per cent. more; thus his income exceeds that of the first person by £80. A third person has £500 more capital than the second, and invests it one per cent. more advantageously; and thus receives £70 more income. Find the capital of each and the rate of investment.

4. Find four numbers, such that the first with half the rest, the second with a third the rest, the third with a fourth the rest, and the fourth with a fifth of the rest shall each be equal to *a*.

5. A number is represented by 6 digits, of which the left-hand digit is 1. If the 1 be removed to units place, the others remaining in the same order as before, the new number is 3 times the original number. Find the number.

6. A man has £22 14*s.* in crowns (5*s.*), guineas (21*s.*), and moidores (27*s.*); and he finds that if he had as many guineas as crowns, and as many crowns as guineas, he would have £36 6*s.*; but if he had as many crowns as moidores, and as many moidores as crowns, he would have £45 16*s.* How many of each has he?

7. A person has four casks, the second of which being filled from the first, leaves the first $\frac{4}{7}$ full. The third being filled from the second, leaves it $\frac{1}{4}$ full; and when the third is emptied into the fourth, it is found to fill only $\frac{9}{16}$ of it. But the first will fill the third and fourth and have fifteen quarts remaining. How many quarts does each hold?

8. A, B, C, and D, engage to do a certain work. A and B can do it in 12 days, A and D in 15 days, and D and C in 18 days. B and C commence the work together, after 3 days are joined by A, and after 4 days more by D. Then, all working together, they finish it in 2 days. How long would each have required to do the entire work? Solve with one unknown quantity, as well as with four.

9. A person sculling in a thick fog, meets one tug and overtakes another which is going at the same rate as the former; show that if a is the greatest distance to which he can see, and b, b' are the distances that he sculls between the times of his first seeing and of his passing the tugs, $\frac{2}{a} = \frac{1}{b} + \frac{1}{b'}$.

CHAPTER II.

RATIO, PROPORTION, AND PROGRESSION.

SECTION I.

RATIO.

44. *Ratio* is the relative magnitude of one quantity as compared with another of the same kind, and is expressed by the quotient arising from dividing the first by the second.* The first quantity named is called the *Antecedent*, and the second the *Consequent*. Taken together they are called the *Terms* of the ratio, or a *Couplet*.

45. *The Sign* of ratio is the colon, :, the common sign of division, ÷, or the fractional form of indicating division.

The last form is coming into use quite generally, and is to be preferred.

46. Cor.—*A ratio being merely a fraction, or an unexecuted problem in Division, of which the antecedent is the numerator, or dividend, and the consequent the denominator, or divisor, any changes made upon the terms of a ratio produce the same effect upon its value, as the like changes do upon the value of a fraction, when made upon its corresponding terms. The principal of these are,*

1st. *If both terms are multiplied, or both divided by the same number, the value of the ratio is* NOT CHANGED.

2d. *A ratio is* MULTIPLIED *by multiplying the antecedent, or by dividing the consequent.*

3d. *A ratio is* DIVIDED *by dividing the antecedent, or by multiplying the consequent.*

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* There is a common notion among us that the French express a ratio by dividing the consequent by the antecedent, while the English express it as above. Such is not the fact. French, German, and English writers agree in the above definition. In fact, the Germans very generally use the sign : instead of ÷; and by all, the two signs are used as exact equivalents.

47. ***A Direct Ratio*** is the quotient of the antecedent divided by the consequent, as explained above, (**44**). An ***Indirect*** or ***Reciprocal Ratio*** is the quotient of the consequent divided by the antecedent, *i. e.*, the *reciprocal* of the direct ratio. *A ratio is always* WRITTEN *as a direct ratio.*

48. A ratio of ***Greater Inequality*** is a ratio which is greater than unity, as $4 : 3$. A ratio of ***Less Inequality*** is a ratio which is less than unity, as $3 : 4$.

49. ***A Compound Ratio*** is the product of the corresponding terms of several simple ratios. Thus, the compound ratio $a : b$, $c : d$, $m : n$, is $acm : bdn$. This term corresponds to *compound fraction*. A compound ratio is the same in effect as a compound fraction.

50. ***A Duplicate Ratio*** is the ratio of the *squares*, a ***triplicate,*** of the *cubes*, a ***subduplicate,*** of the *square roots*, and a ***subtriplicate,*** of the *cube roots* of two numbers. Thus, $a^2 : b^2$, $a^3 : b^3$, $\sqrt{a} : \sqrt{b}$, and $\sqrt[3]{a} : \sqrt[3]{b}$.

EXAMPLES.

1. What is the ratio of 8 to 4? of 4 to 8? of $\frac{1}{2}$ to $\frac{2}{3}$? of $5a^2m$ to $3am$? of $x^2 - y^2$ to $x - y$? of $\frac{3}{4}$ to $\frac{4}{3}$? of $\frac{m}{n}$ to $\frac{a}{b}$? of $\frac{a^2 - b^2}{1 - x}$ to $\frac{a + b}{1 - x}$?

2. Write the inverse ratio in each case in the last paragraph.

3. Reduce the following to their lowest terms: $85 : 187$, $a^2 - b^2 : a^4 - b^4$, $12(a - x)^2 : 6(a^2 - x^2)$.

4. What is the duplicate ratio of $3 : 5$, of $a : b$? What the triplicate? What the subduplicate of $25 : 16$? of $3 : 7$? of $m : n$? What the subtriplicate of $729 : 1728$? of $x : y$?

5. Which is the greater, the compound ratio of $\frac{2}{3} : \frac{3}{5}$ and $5 : 4$, or the inverse triplicate ratio of $3 : 2$?

6. Prove that a ratio of greater inequality is diminished by adding the same number to both its terms. How is it with a ratio of less inequality? How with equality?

7. If 5 gold coins and 30 silver ones are worth as much as 10 gold coins and 10 silver ones, what is the ratio of their values?

8. Prove that $a^2 - x^2 : a^2 + x^2 > a - x : a + x$. Is $x^3 + y^3 : x^2 + y^2$ greater, or less, than $x^2 + y^2 : x + y$?

9. Prove that $4a^3 - 3a^2x - 4ax^2 + 3x^3 : 3a^3 - 2a^2x - 3ax^2 + 2x^3$ is equal to $4a - 3x : 3a - 2x$.

10. Prove that, if x be to y in the duplicate ratio of a to b, and a to b in the subduplicate ratio of $a + x$ to $a - y$, then will $2x : a = x - y : y$.

SECTION II.

PROPORTION.

51. Proportion is an equality of ratios, the terms of the ratios being expressed. The equality is indicated by the ordinary sign of equality, $=$, or by the double colon, $::$.

SCH.—The pupil should practice writing a proportion in the form $\frac{a}{b} = \frac{c}{d}$, still *reading* it "$a$ is to b as c is to d." One form should be as familiar as the other. He must accustom himself to the thought that $a : b :: c : d$ means $\frac{a}{b} = \frac{c}{d}$ and *nothing more.*

52. The Extremes (outside terms) of a proportion are the first and fourth terms. The ***Means*** (middle terms) are the second and third terms.

53. A Mean Proportional between two quantities is a quantity to which either of the other two bears the same ratio that the mean does to the other of the two.

54. A Third Proportional to two quantities is such a quantity that the first is to the second as the second is to this third (proportional).

55. A proportion is taken by ***Inversion*** when the terms of each ratio are written in inverse order.

56. A proportion is taken by ***Alternation*** when the means are made to change places, or the extremes.

57. A proportion is taken by ***Composition*** when the sum of the terms of each ratio is compared with either term of that ratio, the same order being observed in both ratios; or when the sum of the antecedents and the sum of the consequents are compared with either antecedent and its consequent.

58. If the *difference* instead of the *sum* be taken in the last definition, the proportion is taken by ***Division.***

59. Four quantities are *Inversely* or *Reciprocally Proportional* when the first is to the second as the fourth is to the third, or as the reciprocal of the third is to the reciprocal of the fourth.

60. A Continued Proportion is a succession of equal ratios, in which each consequent is the antecedent of the next ratio. Thus if $a:b::b:c::c:d::d:e$, we have a continued proportion.

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61. Prop. 1.—*In any proportion the product of the extremes equals the product of the means.*

DEM.—If $a:b::c:d$ then $ad=bc$. For $a:b::c:d$ is the same as $\frac{a}{b}=\frac{c}{d}$, which cleared of fractions becomes $ad=bc$. Q. E. D.

62. COR. 1.—*The square of a mean proportional equals the product of its extremes, and hence a mean proportional itself equals the square root of the product of its extremes.*

If $a:m::m:d$, by the proposition $m^2=ad$. Whence extracting the square root of both members, $m=\sqrt{ad}$.

63. COR. 2.—*Either extreme of a proportion equals the product of the means divided by the other extreme; and, in like manner, either mean equals the product of the extremes divided by the other mean.*

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64. Prop. 2.—*If the product of two quantities equals the product of two others, the two former may be made the extremes, or the means of a proportion, and the two latter the other terms.*

DEM.—Suppose $my=nx$. Dividing both members by xy, we have $\frac{m}{x}=\frac{n}{y}$, *i. e.*, $m:x::n:y$. In like manner dividing by mn we have $\frac{y}{n}=\frac{x}{m}$, *i. e.*, $y:n::x:m$.

Deduce each of the following forms from the relation $my=nx$:

1. $m:x::n:y$.
2. $m:n::x:y$.
3. $y:n::x:m$.
4. $x:y::m:n$.
5. $y:x::n:m$.
6. $x:m::y:n$.
7. $n:m::y:x$.
8. $n:y::m:x$.

65. Cor.—*If four quantities are in proportion, they are in proportion by alternation and by inversion.*

---

66. Prop. 3.—*If four quantities are in proportion, the proportion is not destroyed by taking equal multiples of*

1st. *The terms of the same couplet,*

2d. *The antecedents,*

3d. *The consequents,*

4th. *All the terms.*

Demonstrate these facts from the nature of a proportion as an equality of ratios.

67. Sch.—Observe that such changes, and only such, may be made upon the terms of a proportion without destroying it, as

1st. *Do not change the values of the ratios,*

2d. *Change both ratios alike.*

Query.—If the first term of a proportion be divided by any number, in what ways may the operation be compensated for so as to preserve the proportion?

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68. Prop. 4.—*The products or the quotients of the corresponding terms of two (or more) proportions are proportional to each other.*

Demonstrated on the axioms that equals multiplied by equals give equal products, and that equals divided by equals give equal quotients.

69. Cor.—*Like powers, or roots, of proportionals are proportional to each other.*

How does this corollary grow out of the proposition?

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70. Prop. 5.—*If two proportions have a ratio in one equal to a ratio in the other, the remaining ratios are equal and may form a proportion.*

Demonstrated on the axiom that things which are equal to the same thing are equal to each other.

---

71. Prop. 6.—*Any proportion may be taken by composition, or by division, or by both at once, without destroying it.*

DEM.—If $a : b :: c : d$,

We may write by composition,

$$\begin{cases} a + b : b :: c + d : d, & (1) \\ a + b : a :: c + d : c, & (2) \\ a + c : a :: b + d : b, & (3) \\ a + c : c :: b + d : d. & (4) \end{cases}$$

By division, we may write the same forms with the $-$ sign instead of the $+$.

By composition and division at the same time, we may write,

$$\begin{cases} a + b : a - b :: c + d : c - d, \\ a + c : a - c :: b + d : b - d. \end{cases}$$

These forms may all be verified by representing the ratio of a to b by r, whence $\frac{a}{b} = r$, or $a = rb$, and since the ratio of c to d is the same as that of a to b, $\frac{c}{d} = r$, or $c = dr$, and then substituting in each of the above forms these values of a and c. Thus, the 1st becomes $br + b : b :: dr + d : d$, which ratios are equal, since each is $r + 1$. Let the student verify the other forms in the same way.

QUERIES.—If $a : b :: c : d$, is $a \pm b : a :: c \pm d : b$? Is $a + b : c + d :: a - c : b - d$?

72. COR.—*If there be a series of equal ratios in the form of a continued proportion, the sum of all the antecedents is to the sum of all the consequents, as any one antecedent is to its consequent.*

DEM.—If $a : b :: c : d :: e : f :: g : h$, etc., $a + c + e + g +$ etc. $: b + d + f + h +$ etc. $:: a : b$, or $c : d$, or $e : f$, or $g : h$, etc. Substitute for a br, for c dr, for e fr, for g hr, and we have

$$br + dr + fr + hr + \text{etc.} : b + d + f + h + \text{etc.} :: br : b,$$

in which the ratios are seen to be equal, since each is r.

73. SCH.—*The method pursued in the demonstration of the preceding proposition will be found sufficient in itself to test any proposed transformation of a proportion.* We will give a few examples:

1. If $a : b :: c : d$, prove as above that $ad = bc$.

SUG.—By substituting as above we have the identity $brd = bdr$.

2. If $a : b :: c : d$, prove as above that $a : c :: b : d$, and $b : a :: d : c$.

3. If $a : b :: c : d$, and $m : n :: x : y$, prove as above that $am : bn :: cx : dy$.

SUG'S.—Let $\frac{a}{b} = r$, whence $\frac{c}{d} = r$; and $\frac{m}{n} = r'$, whence $\frac{x}{y} = r'$. Substituting for a br, for c dr, for m nr', and for x yr', in the proportion to be tested, it is shown to be true.

4. If $\frac{1}{2}a - x : \frac{1}{2}a + x :: b - y : b + y$, show that $2x : y :: a : b$.

SUG'S.—From $\frac{\frac{1}{2}a - x}{\frac{1}{2}a + x} = r$ find x in terms of a and r, and from $\frac{b - y}{b + y} = r$ find y in terms of b and r.

5. If $a : b :: p : q$, prove that $a^2 + b^2 : \frac{a^3}{a + b} :: p^2 + q^2 : \frac{p^3}{p + q}$.

6. Four given numbers are represented by a, b, c, d; what quantity added to each will make them proportionals?

Ans., $\frac{bc - ad}{a - b - c + d}$.

7. If four numbers are proportionals, show that there is no number which, being added to each, will leave the resulting four numbers proportionals.

8. If $a : b :: c : d$, show that $ma : mb :: c : d$; $a : b :: mc : md$; $ma : b :: mc : d$; $a : mb :: c : md$; and $ma : nb :: mc : nd$.

APPLICATIONS.

[NOTE.—The first five of the following examples should be solved without converting the proportions into equations.]

1. A merchant having mixed a certain number of gallons of brandy and water, found that if he had mixed 6 gallons more of each, there would have been 7 gallons of brandy to every 6 gallons of water, but, if he had mixed 6 gallons less of each, there would have been 6 gallons of brandy to every 5 gallons of water. How much of each did he mix?

SOLUTION. $x + 6 : y + 6 :: 7 : 6$, and $x - 6 : y - 6 :: 6 : 5$.
Hence $x - y : y + 6 :: 1 : 6$, and $x - y : y - 6 :: 1 : 5$.
Hence $y + 6 : y - 6 :: 6 : 5$, or $2y : 12 :: 11 : 1$, or $y : 66 :: 1 : 1$.
Substituting, $x + 6 : 72 :: 7 : 6$, or $x + 6 : 6 :: 14 : 1$, or $x : 6 :: 13 : 1$, or $x : 1 :: 78 : 1$.

2. Find two numbers, such, that their sum, difference, and product, may be as the numbers s, d, and p, respectively.

SOLUTION. $x + y : x - y :: s : d$, and $x - y : xy :: d : p$.
Hence $x : y :: s + d : s - d$, and $x : y :: dx + p : p$.
Hence $dx + p : p :: s + d : s - d$, or $dx : p :: 2d : s - d$, or $x : p :: 2 : s - d$,

or $x : 1 :: 2p : s - d$, *i. e.* $x = \frac{2p}{s - d}$.

3. It is required to find a number, such, that the sum of its digits is to the number itself as 4 to 13; and if the digits be inverted, their difference will be to the number expressed as 2 to 31.

4. Divide the number 14 into two such parts, that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater, as 16 to 9.

5. Find two numbers whose difference is to the difference of their squares as $m : n$, and whose sum is to the difference of their squares as $a : b$.

[NOTE.—In the following, use the proportion more or less, as is found expedient.]

6. The sides of a triangle are as 3 : 4 : 5, and the perimeter is 480 yards: find the sides.

7. A fox makes 4 leaps while a hound makes 3; but two of the hound's leaps are equivalent to 3 of the fox's. What are their relative rates of running?

8. A courier sets out from Trenton for Washington, and travels at the rate of 8 miles an hour; two hours after his departure another courier sets out after him from New York, supposed to be 68 miles distant from Trenton, and travels at the rate of 12 miles an hour. How far must the second courier travel before he overtakes the first?

9. There are two places, 154 miles apart, from which two persons set out at the same time to meet, one travelling at the rate of 3 miles in two hours, and the other at the rate of 5 miles in four hours. How long, and how far, did each travel before they met?

10. A courier, who travels 60 miles a day, has been dispatched five days, when a second is sent to overtake him, in order to do which he must travel 75 miles a day. In what time will he overtake the former?

11. Two travellers, A and B, set out at the same time from two different places, C and D; A from C to D, and B from D to C. When they met, it appeared that A had gone 30 miles more than B; also, that A can reach D in 4 days, and B can reach C in 9 days. Required the distance from C to D.

12. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's 3; but two of the greyhound's leaps are as much as

three of the hare's. How many leaps must the greyhound take to catch the hare?

13. A runner left this place n days ago, at the rate of a miles daily. He is pursued by another, at the rate of b miles a day. In how many days will the second overtake the first?

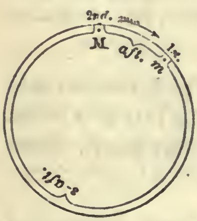
Ans., $\frac{an}{b-a}$.

14. Find the time between 3 and 4 when the hands of a watch are opposite each other. When they are at right angles to each other. When they are together.

15. How often does the minute hand of a watch pass the hour hand? How often at right angles? How often opposite?

16. Do the hands of a watch occupy the three relative positions of opposite, at right angles, and together between each two hours of the 12? If there are exceptions point them out, and show why they occur.

17. Before noon, a clock which is too fast, and points to afternoon time, is put back 5 hours and 40 minutes; and it is observed that the time before shown is to the true time as 29 to 105. Required the true time.



18. Two bodies move uniformly around the circumference of the same circle, which measures s feet. When they start, one is a feet before the other; but the first moves m and the second M feet in a second. When will these bodies pass each other the first time, when the second, when the third, etc., supposing that they do not disturb each other's motion? When will they pass if the first starts t seconds before the second, and $M > m$? When if $M < m$? When will they pass if the first starts t seconds later than the second and $M > m$? When if $M < m$? When will they meet if they start at the same time and move toward each other, or over the distance a, first? If they move from each other, or over the arc $s - a$ first? When will they meet if the first starts t seconds before the other, and they move toward each other, or over the distance a first? If they move from each other, or over the arc $s - a$ first? If they move in opposite directions, and the first starts t seconds later than the second? When they move over the arc a first? When they move over the arc $s - a$ first?

19. The force of gravitation is inversely as the square of the distance from the centre of the earth. At the distance 1 from the centre of the earth this force is expressed by the number 32.16. By what is it expressed at the distance 60 ? *Ans.*, 0.0089.

20. If the velocity of one body moving around another is proportional to unity divided by the duplicate of the distance, and the velocity be represented by v when the distance is r, by what will it be expressed when the distance is r' ?

Ans., $\frac{r^2 v}{r'^2}$.

SECTION III.

PROGRESSIONS.

74. A Progression is a series of terms which increase or decrease by a common difference, or by a common multiplier. The former is called an *Arithmetical*, and the latter a *Geometrical Progression.* A Progression is *Increasing* or *Decreasing* according as the terms increase or decrease in passing to the right. The terms *Ascending* and *Descending* are used in the same sense as increasing and decreasing, respectively. In an Arithmetical Progression the common difference is added to any one term to produce the next term to the right. If the progression is decreasing the common difference is *minus.* In an increasing Geometrical Progression the constant multiplier by which each succeeding term to the right is produced from the preceding is more than unity; and in a decreasing progression it is less than unity. This constant multiplier in a Geometrical Progression is called the *Ratio* of the series.

75. The character, $\cdot\cdot$, is used to separate the terms of an Arithmetical Progression, and the colon, :, for a like purpose in a Geometrical Progression.

ILLUSTRATIONS.

$1 \cdot\cdot 3 \cdot\cdot 5 \cdot\cdot 7$, etc., etc., is an increasing Arithmetical Progression with a common difference 2, or + 2.

$15 \cdot\cdot 10 \cdot\cdot 5 \cdot\cdot 0 \cdot\cdot -5$, etc., etc., is a Decreasing Arithmetical Progression with a common difference -5.

$a \cdot\cdot a \pm d \cdot\cdot a \pm 2d \cdot\cdot a \pm 3d$, etc., etc., is the general form of an Arithmetical Progression, $\pm d$ being the common difference.

$2 : 4 : 8 : 16$, etc., etc., is an increasing Geometrical Progression with ratio 2.

$12 : 4 : \frac{4}{3} : \frac{4}{9} : \frac{4}{27}$, etc., etc., is a Decreasing Geometrical Progression with ratio $\frac{1}{3}$.

$a : ar : ar^2 : ar^3 : ar^4$, etc., etc., is the general form of a Geometrical Progression, r being the ratio, and greater or less than unity, according as the series is increasing or decreasing.

76. When three quantities taken in order are in arithmetical progression, the second is the *Arithmetical Mean* between the other two, and is equal to half their sum.

ILL.—If $a \cdot\cdot b \cdot\cdot c$, b is the arithmetical mean between a and c; and since $b - a = c - b$, $b = \frac{1}{2}(a + c)$.

77. When three quantities taken in order are in geometrical progression, the second is the *Geometric Mean* between the other two, and is equal to the square root of their product.

Let the student illustrate.

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78. There are *Five Things* to be considered in any progression; viz., the first term, the last term, the common difference or the ratio, the number of terms, and the sum of the series, any three of which being given the other two can be found, as will appear from the subsequent discussion.

ARITHMETICAL PROGRESSION.

79. Prop. 1.—*The formula for finding the* n*th, or last term of an Arithmetical Progression; or, more properly, the formula expressing the relation between the first term, the* n*th term, the common difference, and the number of terms of such a series, is*

$$l = a + (n - 1)d,$$

in which a *is the first term,* d *the common difference,* n *the number of terms, and* l *the* n*th or last term,* d *being positive or negative according as the series is increasing or decreasing.*

DEM.—According to the notation, the series is

$$a \cdot\cdot a + d \cdot\cdot a + 2d \cdot\cdot a + 3d \cdot\cdot a + 4d \cdot\cdot a + 5d, \text{ etc., etc.}$$

Hence we observe that as each succeeding term is produced by adding the common difference to the preceding, when we have reached the nth term, we shall have added the common difference to the first term $n - 1$ times; that is, the nth term, or $l = a + (n - 1)d$. Q. E. D.

SCH.—As this formula is a simple equation in terms of a, l, n, and d, any one of them may be found in terms of the other three.

80. Prop. 2.—*The formula for the sum of an Arithmetical Progression, or expressing the relation between the sum of the series, the first term, last term, and number of terms, is*

$$s = \left[\frac{a + l}{2}\right]n,$$

s *representing the sum of the series,* a *the first term,* l *the last term, and* n *the number of terms.*

Dem.—If l is the last term of the progression, the term before it is $l-d$, and the one before that $l-2d$, etc. Hence, as $a \cdot\cdot a+d \cdot\cdot a+2d \cdot\cdot a+3d$ - - - - - l represents the series, $l \cdot\cdot l-d \cdot\cdot l-2d \cdot\cdot l-3d$ - - - - - a represents the same series reversed. Now the sum of the first series is

$$s=a+(a+d)+(a+2d)+ \text{ - - - } (l-2d)+(l-d)+l;$$

and reversed $\quad s=l+(l-d)+(l-2d)+ \text{ - - - } (a+2d)+(a+d)+a.$

Adding $\quad 2s=(a+l)+(a+l)+(a+l)+ \text{ - - } (a+l)+(a+l)+(a+l).$

If the number of terms in the series is n, there will be n terms in this sum, each of which is $(a+l)$; hence $2s=(a+l)n$, or $s=\left[\frac{a+l}{2}\right]n$. Q. E. D.

Sch.—This formula being a simple equation in terms of s, a, l, and n, any one of the four can be found in terms of the other three.

81. Cor. 1.—*Formulas*

(1) $\quad l=a+(n-1)d,$ *and*

(2) $\quad s=\left[\frac{a+l}{2}\right]n,$ *being two equations between*

the five quantities, a, l, n, d, *and* s, *any two of these five can be found in terms of the other three.*

82. Cor. 2.—*The formula for inserting a given number of arithmetical means between two given extremes is* $d=\frac{l-a}{m+1}$, *in which* m *represents the number of means. From this* d, *the common difference, being found, the terms can readily be written.*

Dem.—If a is the first term and l the last, and there are m terms between, or m means, there are in all $m+2$ terms. Hence, substituting in the formula $l=a+(n-1)d$, for n, $m+2$, we have $l=a+(m+1)d$. From this $d=\frac{l-a}{m+1}$. Q. E. D.

83. Formulæ in Arithmetical Progression.

[It will afford a good exercise for the student to solve the following cases on review, after having gone through Quadratics; though no importance need be attached to remembering the results, as the fundamental formulas

$$(1)\ l=a+(n-1)d, \quad \text{and} \quad (2)\ s=\left[\frac{a+l}{2}\right]n,$$

are sufficient to resolve all cases.]

NUMBER.	GIVEN.	REQUIRED.	FORMULAS.
1.	$a,\ d,\ n$	l	$l=a+(n-1)d,$
2.	$a,\ d,\ S$	l	$l=-\frac{1}{2}d\pm\sqrt{\{2dS+(a-\frac{1}{2}d)^2\}},$
3.	$a,\ n,\ S$	l	$l=\frac{2S}{n}-a,$
4.	$d,\ n,\ S$	l	$l=\frac{S}{n}+\frac{(n-1)d}{2}.$
5.	$a,\ d,\ n$	S	$S=\frac{1}{2}n\{2a+(n-1)d\},$
6.	$a,\ d,\ l$	S	$S=\frac{l+a}{2}+\frac{l^2-a^2}{2d},$
7.	$a,\ n,\ l$	S	$S=(l+a)\frac{n}{2},$
8.	$d,\ n,\ l$	S	$S=\frac{1}{2}n\{2l-(n-1)d\}.$
9.	$d,\ n,\ l$	a	$a=l-(n-1)d,$
10.	$d,\ n,\ S$	a	$a=\frac{S}{n}-\frac{(n-1)d}{2},$
11.	$d,\ l,\ S$	a	$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2dS},$
12.	$n,\ l,\ S$	a	$a=\frac{2S}{n}-l.$
13.	$a,\ n,\ l$	d	$d=\frac{l-a}{n-1},$
14.	$a,\ n,\ S$	d	$d=\frac{2(S-an)}{n(n-1)},$
15.	$a,\ l,\ S$	d	$d=\frac{l^2-a^2}{2S-l-a},$
16.	$n,\ l,\ S$	d	$d=\frac{2(nl-S)}{n(n-1)}.$
17.	$a,\ d,\ l$	n	$n=\frac{l-a}{d}+1,$
18.	$a,\ d,\ S$	n	$n=\frac{\pm\sqrt{(2a-d)^2+8dS}-2a+d}{2d},$
19.	$a,\ l,\ S$	n	$n=\frac{2S}{l+a},$
20.	$d,\ l,\ S$	n	$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}.$

EXAMPLES.

1. Find the 21st term of $3 \cdot\cdot 7 \cdot\cdot 11 \cdot\cdot$ etc., and the sum of these terms.

2. Find the 24th term of $7 \cdot\cdot 5 \cdot\cdot 3 \cdot\cdot$ etc., and the sum of these terms.

3. Find the nth term of $\frac{1}{3} \cdot\cdot \frac{5}{6} \cdot\cdot \frac{4}{3} \cdot\cdot$ etc., and the sum of the n terms.

4. Find the nth term of $\frac{n-1}{n} \cdot\cdot \frac{n-2}{n} \cdot\cdot \frac{n-3}{n} \cdot\cdot$ etc., and the sum of the n terms.

5. Insert four arithmetical means between 193 and 443.

6. Prove that the sum of n terms of $1 \cdot\cdot 3 \cdot\cdot 5 \cdot\cdot 7 \cdot\cdot$ etc., is to the sum of m terms as $n^2 : m^2$.

7. What is the first term of an arithmetical progression whose 59th term is $-2\frac{1}{2}$, and 60th $-1\frac{3}{4}$? Whose 2d term is $\frac{1}{2}$, and 55th 5.8?

8. How many terms in the progression whose common difference is 3, first term 5, and last term 302?

9. Insert three arithmetical means between m and n.

10. Produce the formula for inserting m arithmetical means between a and b, viz.,

$$a \cdot\cdot \frac{am+b}{m+1} \cdot\cdot \frac{am-a+2b}{m+1} \text{ - - - - - - } \frac{bm-b+2a}{m+1} \cdot\cdot \frac{bm+a}{m+1} \cdot\cdot b.$$

11. If a body falling to the earth descends a feet the first second, $3a$ the second, $5a$ the third, and so on, how far will it fall during the tth second? *Ans.*, $(2t-1)a$.

12. If a body falling to the earth descends a feet the first second, $3a$ the second, $5a$ the third, and so on, how far will it fall in t seconds? *Ans.*, at^2.

GEOMETRICAL PROGRESSION.

84. Prop. 1.—*The formula for finding the* n*th, or last term of a geometrical progression; or, more properly, the formula expressing the relation between the first term, the* n*th term, the ratio, and the number of terms of such a series, is* $l = ar^{n-1}$, *in which* l *is the last, or* n*th term,* a *the first term,* r *the ratio, and* n *the number of terms.*

DEM.—Letting a represent the first term and r the ratio, the series is $a : ar : ar^2 : ar^3 : ar^4$: etc. Whence it appears that any term consists of the first term multiplied into the ratio raised to a power whose exponent is one less than the number of the term. Therefore the nth term, or $l = ar^{n-1}$. Q. E. D.

85. Prop. 2.—*The formula for the sum of a geometrical progression, or expressing the relation between the sum of the series, the first term, the ratio, and the number of terms is*

$$s = \frac{ar^n - a}{r - 1},$$

in which s *represents the sum,* a *the first term,* r *the ratio, and* n *the number of terms.*

DEM.—The sum of the series being found by adding all its terms, we have,

$$s = a + ar + ar^2 + ar^3 + \cdots ar^{n-3} + ar^{n-2} + ar^{n-1}, \text{ and multiplying by } r,$$
$$rs = \quad ar + ar^2 + ar^3 + \cdots ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n. \text{ Subtracting,}$$

$$rs - s = ar^n - a, \text{ or}$$

$$(r-1)s = ar^n - a, \text{ and } s = \frac{ar^n - a}{r - 1}. \quad \text{Q. E. D.}$$

86. COR. 1.—*Formulas*

$$(1) \qquad l = ar^{n-1}, \qquad \textit{and}$$

$$(2) \qquad s = \frac{ar^n - a}{r - 1}$$ *being two equations between the five quantities,* a, l, r, n, *and* s, *are sufficient to determine any* TWO *of them when the others are given.*

87. COR. 2.—*Since* $l = ar^{n-1}$, $lr = ar^n$, *which substituted in* (2) *gives* $s = \frac{lr - a}{r - 1}$; *which formula is often convenient.*

88. COR. 3.—*The formula for inserting* m *geometrical means between* a *and* l *is* $r = \sqrt[m+1]{\frac{l}{a}}$.

89. COR. 4.—*The formula for the sum of an infinite decreasing geometrical progression is* $s = \frac{a}{1 - r}$.

DEM.—Since in a decreasing progression the ratio is less than unity, the last term, ar^{n-1}, is also less than the first term, and numerator and denominator of the value of s, $\frac{lr - a}{r - 1}$, become negative. Hence it is well enough to write the formula for the sum of such a series $s = \frac{a - lr}{1 - r}$, that is, change the signs of both terms of the fraction. Now, if the terms of a series are constantly decreasing, and the number of terms is infinite, we can fix no value, however small, which will not be greater than the last, or than some term which may be reached and passed. Hence we are compelled to call the last term of such a series 0, which makes the formula $s = \frac{a}{1 - r}$. Q. E. D.

90. Geometrical Formulæ.

[In a review, *after the pupil has been through the book,* it will be a good exercise for him to deduce the following formulas from the two fundamental ones. It is not necessary to memorize these.]

NUMBER.	GIVEN.	REQUIRED.	FORMULÆ.
1.	$a,\ r,\ n$	l	$l = ar^{n-1}$,
2.	$a,\ r,\ S$		$l = \frac{a + (r-1)S}{r}$,
3.	$a,\ n,\ S$		$l\,(S - l)^{n-1} - a(S - a)^{n-1} = 0$,
4.	$r,\ n,\ S$		$l = \frac{(r-1)Sr^{n-1}}{r^n - 1}$.
5.	$a,\ r,\ n$	S	$S = \frac{a(r^n - 1)}{r - 1}$,
6.	$a,\ r,\ l$		$S = \frac{rl - a}{r - 1}$,
7.	$a,\ n,\ l$		$S = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$,
8.	$r,\ n,\ l$		$S = \frac{lr^n - l}{r^n - r^{n-1}}$.
9.	$r,\ n,\ l$	a	$a = \frac{l}{r^{n-1}}$,
10.	$r,\ n,\ S$		$a = \frac{(r-1)S}{r^n - 1}$,
11.	$r,\ l,\ S$		$a = rl - (r-1)S$,
12.	$n,\ l,\ S$		$a(S - a)^{n-1} - l(S - l)^{n-1} = 0$.
13.	$a,\ n,\ l$	r	$r = \sqrt[n-1]{\frac{l}{a}}$,
14.	$a,\ n,\ S$		$r^n - \frac{S}{a}r + \frac{S - a}{a} = 0$,
15.	$a,\ l,\ S$		$r = \frac{S - a}{S - l}$,
16.	$n,\ l,\ S$		$r^n - \frac{S}{S - l}r^{n-1} + \frac{l}{S - l} = 0$.
17.	$a,\ r,\ l$	n	$n = \frac{\log l - \log a}{\log r} + 1$,
18.	$a,\ r,\ S$		$n = \frac{\log\,[a + (r-1)S] - \log a}{\log r}$,
19.	$a,\ l,\ S$		$n = \frac{\log l - \log a}{\log\,(S - a) - \log\,(S - l)} + 1$,
20.	$r,\ l,\ S$		$n = \frac{\log l - \log\,[lr - (r-1)S]}{\log r} + 1$.

Examples.

1. In a geometrical progression the first term is 3, the ratio 5, and the number of terms 7. What is the last term? What the sum?

2. Insert 5 geometrical means between 2 and 1458.

3. Find the 11th term of $\frac{1}{64} : \frac{1}{16} : \frac{1}{4}$: etc., and the sum of the 11 terms.

4. Find the 7th term of $-\frac{2}{3} : \frac{1}{3} : -\frac{1}{6}$: etc., and the sum of the 7 terms.

5. Insert 4 geometrical means between $\frac{1}{2}$ and $\frac{243}{64}$.

6. Find the sum of $3 : \frac{1}{3} : \frac{1}{27}$: etc., to infinity. Also of $\frac{1}{2} : -\frac{1}{3}$: etc., to infinity. Also of $.\dot{5}\dot{4}$. Also of $.8\dot{3}\dot{6}$.

7. If a body move 20 miles the first minute, 19 miles the second, $18\frac{1}{20}$ the third, and so on in geometrical progression forever, what is the utmost distance it can reach? *Ans.*, 400 miles

8. What is the distance passed through by a ball, before it comes to rest, which falls from the height of 50 feet, and at every fall rebounds half the distance, the time of ascent equalling the time of descent? *Ans.*, 150.

9. In the preceding problem, suppose the body falls $16\frac{1}{12}$ feet the first second, 3 times as far the next second, and 5 times as far the third second, and so on, how long will it be before it comes to rest?

Ans., $\frac{10}{193}\sqrt{579}(4+3\sqrt{2}) = 10.27657 +$ seconds.

10. Find the sum of the following series:

$\frac{1}{2}-\frac{1}{3}+\frac{2}{9}-\frac{4}{27}+$ etc., to n terms.
$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$ etc., to 10 terms. Also to infinity.
$1\frac{1}{2}+.5+$ etc., to 12 terms. Also to infinity.

11. To find what each payment must be in order to discharge a given principal and interest in a given number of equal payments at equal intervals of time.

Solution.—Let p represent the principal, r the rate per cent., t one of the equal intervals of time, n the number of payments (*i. e.*, nt is the whole time), and x one of the payments.

There will be as many solutions as there are different methods of computing interest on notes upon which partial payments have been made.

1st. *By the United States Court Rule.*—As the payments must exceed the interest in order to discharge the principal, this rule requires that we find the amount of p, for time t, at r per cent. This is done by multiplying by $1 + \frac{rt}{100}$, and gives $p\left(1 + \frac{rt}{100}\right)$. From this subtracting the payment x, the new principal is $p\left(1 + \frac{rt}{100}\right) - x$. Again, finding the amount of this for another period of time, t, and subtracting the second payment,

$$p\left(1 + \frac{rt}{100}\right)^2 - x\left(1 + \frac{rt}{100}\right) - x.$$

In like manner, after the third payment there remains

$$p\left(1 + \frac{rt}{100}\right)^3 - x\left(1 + \frac{rt}{100}\right)^2 - x\left(1 + \frac{rt}{100}\right) - x.$$

After the 4th payment, the remainder is

$$p\left(1 + \frac{rt}{100}\right)^4 - x\left(1 + \frac{rt}{100}\right)^3 - x\left(1 + \frac{rt}{100}\right)^2 - x\left(1 + \frac{rt}{100}\right) - x.$$

Finally, after the nth payment, we have

$$p\left(1 + \frac{rt}{100}\right)^n - x\left(1 + \frac{rt}{100}\right)^{n-1} - x\left(1 + \frac{rt}{100}\right)^{n-2} - \cdots \cdots x\left(1 + \frac{rt}{100}\right)^2$$

$$- x\left(1 + \frac{rt}{100}\right) - x = 0.$$

Whence

$$x = \frac{p\left(1 + \frac{rt}{100}\right)^n}{1 + \left(1 + \frac{rt}{100}\right) + \left(1 + \frac{rt}{100}\right)^2 + \left(1 + \frac{rt}{100}\right)^3 \cdots \cdots \left(1 + \frac{rt}{100}\right)^{n-1}}.$$

This denominator being the sum of a geometrical progression whose first term is 1, ratio $\left(1 + \frac{rt}{100}\right)$, and number of terms n, its sum is $\frac{\left(1 + \frac{rt}{100}\right)^n - 1}{\frac{rt}{100}}$.

Hence $x = \dfrac{\frac{prt}{100}\left(1 + \frac{rt}{100}\right)^n}{\left(1 + \frac{rt}{100}\right)^n - 1}$.

2d. *By the Vermont Rule.*—The amount of the principal for the whole time is $p\left(1 + \frac{rtn}{100}\right)$.

The amount of the 1st payment is - - - - - - - - - - $x\left[1 + \frac{rt}{100}(n-1)\right]$,

" " " 2d " - - - - - - - - - - $x\left[1 + \frac{rt}{100}(n-2)\right]$,

" " " 3d " - - - - - - - - - - $x\left[1 + \frac{rt}{100}(n-3)\right]$,

etc., etc., - - - - - - - - - - etc.

" " - - - - - - - - - - "

The nth payment (with no interest) is - - - - - - - - - - x.

The sum of the amounts of these payments is

$$nx + \frac{rt}{100}x\,[(n-1) + (n-2) + (n-3) - - - - - 1].$$

The series in the brackets being an arithmetical progression whose first term is $(n-1)$, common difference -1, last term 1, and number of terms $(n-1)$, its sum is $\left(\frac{n-1}{2}\right)n$. Hence the sum of the payments is $nx + \frac{rt}{100}x\left(\frac{n-1}{2}\right)n$, or $x\left[n + \frac{\frac{nrt}{100}(n-1)}{2}\right]$. But by the condition this sum equals the amount of the principal; consequently

$$x\left[n + \frac{\frac{nrt}{100}(n-1)}{2}\right] = p\left(1 + \frac{nrt}{100}\right), \text{ and } \quad x = \frac{2p\left(1 + \frac{nrt}{100}\right)}{2n + \frac{nrt}{100}(n-1)}.$$

SCH.—If the payments are made annually, $t = 1$. And letting $r' = \frac{r}{100}$, *i. e.*, letting the rate per cent. be expressed decimally, the formulas become,

By the U. S. Rule, $\quad x = \frac{pr'(1 + r')^n}{(1 + r')^n - 1}$;

By the Vermont Rule, $\quad x = \frac{2p(1 + r'n)}{2n + r'n(n-1)}$.

12. What must be the annual payment in order to discharge a note of \$5000, bearing interest at 10% per annum, in 5 equal payments?

Ans., By the U. S. Rule, \$1318.99 within a half cent.
By the Vermont Rule, \$1250.

QUERY.—What occasions the great disparity between the payments required by the different rules?

SECTION IV.

VARIATION.

91. *Variation* is a term applied to the consideration of quantities so related to each other that any change in one makes the others change in the same ratio, direct or inverse.

One quantity varies *directly* as another, when any change in the latter makes the former change in *the same* (*direct*) ratio.

One quantity varies *inversely* as another, when any change in the latter makes the former change in the corresponding *inverse* ratio.

ILL'S.—The amount earned by a laborer in a given time varies *directly* as his daily wages. The time required to earn a given amount varies *inversely* as the daily wages.

92. One quantity varies *jointly* as two others, when any change in the product of the latter two makes the former change in the same ratio as this product.

ILL.—The amount a laborer receives varies jointly as his daily wages and the time of service.

93. One quantity varies *directly* as a *second* and *inversely* as a *third*, when it varies as the quotient of the second divided by the third.

ILL.—The time required to earn any amount varies *directly* as the amount, and *inversely* as the daily wages.

94. *The Sign* of variation is $\propto$.

ILL.—If x varies directly as y, we write $x \propto y$, and read "x varies as y." If x varies *inversely* as y, we write $x \propto \frac{1}{y}$, and read "x varies inversely as y." If x varies *jointly* as y and z, we write $x \propto yz$, and read "x varies jointly as y and z." If x varies *directly* as y and inversely as z, we write $x \propto \frac{y}{z}$, and read "x varies directly as y, and inversely as z."

95. *Prop.*—*Variation may always be expressed in the form of a proportion.*

DEM.—1st. The expression $x \propto y$ signifies that if x is doubled y is doubled, if x is divided y is divided by the same number, etc.; *i. e.*, that the ratio of x to y is constant. Let m be this ratio, so that $\frac{x}{y} = m$. Therefore $x : y :: m : 1$.

2d. The expression $x \propto \frac{1}{y}$ signifies that if y is multiplied by any number, x is divided by the same, and if y is divided by any number x is multiplied by the same. Hence the product of x and y is constant. Let this product be m. Then $xy = m$, or $x : 1 :: m : y$.

3d. $x \propto yz$ signifies that the ratio of x to yz is constant. Let this be m. Then $\frac{x}{yz} = m$, or $x : yz :: m : 1$, or $x : y :: mz : 1$, or $x : z :: my : 1$, or $x : y :: z : \frac{1}{m}$.

4th. $x \propto \frac{y}{z}$ signifies that the ratio of x to $\frac{y}{z}$ is constant. Let this be m. Then $x : \frac{y}{z} :: m : 1$, or $x : y :: m : z$

Exercises.

1. If $x \propto y$, and $y \propto z$, show that $x \propto z$.

Dem.—If $x \propto y$, the ratio of x to y is constant. Let this ratio be m. Then $x = my$. In like manner let n be the ratio of y to z. Then $y = nz$. Hence $x = mnz$. That is, the ratio of x to z is constant, or $x \propto z$.

2. If $x \propto \frac{1}{y}$, and $y \propto \frac{1}{z}$, show that $x \propto z$.

Sug's.—We may write $x = \frac{m}{y}$, and $y = \frac{n}{z}$. Hence $x = \frac{m}{n}z$. That is, the ratio of x to z is constant, or $x \propto z$.

3. If $x \propto z$, and $y \propto \frac{1}{z}$, show that $x \propto \frac{1}{y}$.

Sug's. $x = mz$, $y = \frac{n}{z}$, $\therefore x = \frac{mn}{y}$, or $x \propto \frac{1}{y}$.

4. If $x \propto y$, show that $\frac{x}{z} \propto \frac{y}{z}$, and $xz \propto yz$.

5. If $x \propto y$, and $z \propto u$, show that $xz \propto yu$, and $\frac{x}{z} \propto \frac{y}{u}$.

6. If $x \propto y$, and $y^2 \propto z^2$, how does x vary in respect to z?

7. If $x \propto y$, and for $x = 8$, $y = 4$, what is the value of y for $x = 20$?

Solution.—Since $x \propto y$, and for $x = 8, y = 4$, the ratio of x to y is 2. That is, $\frac{x}{y} = 2$. Hence for $x = 20$, we have $\frac{20}{y} = 2$, or $y = 10$.

8. If $x \propto \frac{1}{y}$, and for $x = 6$, $y = 2$, what is the value of x for $y = 3$?

Sug. $x : \frac{1}{y} :: 6 : \frac{1}{2}$. Hence for $y = 3$, $x = 4$. Or we may reason thus, in changing from 2 to 3, y increases $\frac{3}{2}$ times. Then, as x changes in the reciprocal ratio, $x = \frac{2}{3}$ of $6 = 4$.

9. If $a + b \propto a - b$, prove that $a^2 + b^2 \propto ab$.

10. If $y = p + q$, in which $p \propto x$ and $q \propto \frac{1}{x}$; and if when $x = 1$, $y = 6$; and when $x = 2$, $y = 5$; prove that $y = \frac{4}{3}x + \frac{14}{3x}$.

11. The area of a triangle equals half the product of the base and altitude. Show that if the base is constant the area varies as the altitude; if the altitude is constant the area varies as the base; and if the area is constant the altitude and base vary inversely.

12. The volume of a pyramid varies jointly as its base and altitude. A pyramid whose base is 9 feet square, and height 10 feet, contains 10 cubic yards. What must be the height of a pyramid with a base 3 feet square in order that it may contain 2 cubic yards?

13. Given that $s \propto t^2$, when f is constant; and $s \propto f$, when t is constant; also, $2s = f$, when $t = 1$. Find the equation between f, s, and t.

SUG.—The first two conditions are equivalent to saying that s varies jointly as t^2 and f, *i. e.* $s \propto ft^2$; since in this expression if f is constant $s \propto t^2$, and if t is constant $s \propto f$.

SECTION V.

HARMONIC PROPORTION AND PROGRESSION.

96. Three quantities are in *Harmonic* Proportion when the difference between the first and second is to the difference between the second and third (the differences being taken in the same order) as the first is to the third.

ILL. 6, 4, and 3 are in harmonic proportion, since $6 - 4 : 4 - 3 :: 6 : 3$. If a, b, c are in harmonic proportion, $a - b : b - c :: a : c$.

97. DEF.—When three quantities taken in order are in harmonic proportion, the second is the *Harmonic Mean* between the other two.

98. ***Prop.***—*If three quantities are in harmonic proportion, their reciprocals are in arithmetical proportion.*

DEM.—If a, b, c are in harmonic proportion, $a - b : b - c :: a : c$, and $ac - bc = ab - ac$. Dividing by abc, we have $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$, *i. e.* $\frac{1}{a} \cdot\cdot \frac{1}{b} \cdot\cdot \frac{1}{c}$.

99. DEF.—The reciprocals of the terms of an arithmetical progression form what is called a *Harmonic Progression.*

ILL.—Thus as 1, 2, 3, 4, 5, 6 is an arithmetical progression, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ is a harmonic progression. Also if a, b, c, d, etc., constitute a harmonic progression, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, etc., constitute an arithmetical progression.

100. SCH.—The term Harmonic is applied to such a series, since, if strings of the same size, substance, and tension, be taken of the lengths $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, any two of them vibrating together produce harmony of sound.

EXERCISES.

1. If a, b, c, d are in harmonic progression, show that $ab : cd :: a - b : c - d$.

SUG'S. $\frac{1}{a} \cdot\cdot \frac{1}{b} \cdot\cdot \frac{1}{c} \cdot\cdot \frac{1}{d}$. Hence $\frac{1}{b} - \frac{1}{a} = \frac{1}{d} - \frac{1}{c}$, or $acd - bcd = abc - abd$.

2. If a, b, c are in harmonic proportion, show that b (the harmonic mean) $= \frac{2ac}{a + c}$.

3. Show that the geometric mean between two numbers is a geometric mean between their arithmetic and harmonic means.

4. To insert n harmonic means between a and b.

SUG.—First find the form of the terms for n arithmetical means between $\frac{1}{a}$ and $\frac{1}{b}$. See (*82*). The harmonic series is $a, \frac{ab(n + 1)}{bn + a}, \frac{ab(n + 1)}{bn + 2a - b}, \cdots \frac{ab(n + 1)}{an + b}, b$.

5. If a and b are the first two terms of a harmonic progression, show that the nth term is $\frac{ab}{a(n - 1) - b(n - 2)}$.

6. Insert 3 harmonic means between $\frac{1}{5}$ and $\frac{1}{17}$.

SCH.—There is no method known for finding the sum of a harmonic series.

CHAPTER III.

QUADRATIC EQUATIONS.

SECTION I.

PURE QUADRATICS.

101. A Quadratic Equation is an equation of the second degree (**6, 8**).

102. Quadratic Equations are distinguished as *Pure* (called also *Incomplete*), and *Affected* (called also *Complete*).

103. A Pure Quadratic Equation is an equation which contains no power of the unknown quantity but the second; as $ax^2 + b = cd$, $x^2 - 3b = 102$.

104. An Affected Quadratic Equation is an equation which contains terms of the second degree and also of the first, with respect to the unknown quantity or quantities; as $x^2 - 4x = 12$, $5xy - x - y^2 = 16a$, $mxy + y = b$.

105. A Root of an equation is a quantity which substituted for the unknown quantity satisfies the equation.

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106. Prob.—*To solve a Pure Quadratic Equation.*

RULE.—TRANSPOSE ALL THE TERMS CONTAINING THE UNKNOWN QUANTITY INTO THE FIRST MEMBER, AND UNITE THEM INTO ONE, CLEARING OF FRACTIONS IF NECESSARY. TRANSPOSE THE KNOWN TERMS INTO THE SECOND MEMBER. DIVIDE BY THE COEFFICIENT OF THE UNKNOWN QUANTITY. FINALLY, EXTRACT THE SQUARE ROOT OF BOTH MEMBERS.

DEM.—According to the definition of a Pure Quadratic, all the terms containing the unknown quantity contain its square. Hence they can be transposed and united into one by adding with reference to the square of the unknown quantity. That transposition, and division of both members by the same quantity, do not destroy the equality has already been proved. Extracting the square root of the first member gives the first power of the unknown quantity, *i. e.* the quantity itself. And taking the square root of both members does not destroy the equation, since like roots of equal quantities are equal.

107. Cor. 1.—*Every Pure Quadratic Equation has two roots numerically equal but with opposite signs.*

For every such equation, as the process of solution shows, can be reduced to the form $x^2 = a$ (a representing any quantity whatever). Whence, extracting the root, we have $x = \pm \sqrt{a}$; as the square root of a quantity is both +, and − (**203,** Part I).

108. Cor. 2.—*The roots of a Pure Quadratic Equation may both be imaginary, and* BOTH *will be if* ONE *is.*

For if after having transposed and reduced to the form $x^2 = a$, the second member is negative, as $x^2 = -a$, extracting the square root gives $x = +\sqrt{-a}$, and $x = -\sqrt{-a}$, both imaginary.

Examples.

1. $5\frac{1}{4}x^2 - 18x + 65 = (3x-3)^2$.

2. $5x^2 - 9 = 2x^2 + 66$.

3. $\dfrac{a}{x} + \dfrac{\sqrt{a^2-x^2}}{x} = \dfrac{x}{b}$.

4. $\dfrac{45}{2x^2+3} = \dfrac{57}{4x^2-5}$.

5. $\dfrac{1}{\sqrt{a-x}+\sqrt{a}} + \dfrac{1}{\sqrt{a+x}-\sqrt{a}} = \dfrac{\sqrt{a}}{x}$.

6. $\dfrac{x^2-12}{3} = \dfrac{x^2-4}{4}$.

7. $x^2 - ax + b = ax(x-1)$.

8. $8 + 3x^2 = 5 + 2x^2$.

9. $\sqrt{\dfrac{a^2}{x^2}+b^2} - \sqrt{\dfrac{a^2}{x^2}-b^2} = b$.

10. $\dfrac{2+x}{4+9n} = \dfrac{1}{2-x}$.

11. $12 + 4(x^2+12) = (2-x)(2+x) - 16$.

12. $x\sqrt{6+x^2} = 1 + x^2$.

13. $\dfrac{ax+1+\sqrt{a^2x^2-1}}{ax+1-\sqrt{a^2x^2-1}} = \frac{1}{2}bx$.

14. $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}} = b$.

Applications.

1. Find two numbers which shall be to each other as 3 to 5, and the difference of whose squares shall be 256.

2. Find a number such that if the square root of the difference between the square of the number and a^2, be successively subtracted from and added to a, the difference of the reciprocals of these results shall be equal to a divided by the square of the number.

3. Find three numbers which shall be to each other as m, n, and p, and the sum of whose squares shall be s.

4. An army was drawn up with 5 more men in file than in rank, but when the form was changed so that there were 845 more in rank, there were but 5 ranks. How many men were there in the army?

5. From two towns, m miles asunder, two persons, A and B, set out at the same time, and met each other, after travelling as many days as are equal to the difference of miles they travelled per day, when it appeared that A had travelled n miles. How many miles did each travel per day?

6. For comparatively small distances above the earth's surface the distances through which bodies fall under the influence of gravity are as the squares of the times. Thus, if one body is falling 2 seconds and another 3, the distances fallen through are as 4 : 9. A body falls 4 times as far in 2 seconds as in 1, and 9 times as far in 3 seconds. These facts are learned both by observation and theoretically. It is also observed that a body falls $16\frac{1}{12}$ feet in 1 second. How long is a body in falling 500 feet? One mile (5280 ft.)? Five miles?

7. The mass of the earth is to the mass of the sun as 1 : 354936, and attraction varies directly as the mass and inversely as the square of the distance. The distance between the earth's centre and sun's centre being 91,430,000 miles, find the point between the earth and sun where the attraction of the earth is equal to that of the sun. The earth's radius being 3,962 miles, where is this point situated with reference to the earth's surface?

8. A certain sum of money is lent at 5% per annum. If we multiply the number of dollars in the principal by the number of dollars in the interest for 3 months, the product is 720. What is the sum lent?

9. The intensity of two lights, A and B, is as 7 : 17, and their distance apart 132 feet. Where in the line of the lights are the points of equal illumination, assuming that the intensity varies inversely as the square of the distance?

10. The loudness of one church bell is three times that of another. Now, supposing the strength of sound to be inversely as the square of the distance, at what place on the line of the two will the bells be equally well heard, the distance between them being a?

SECTION II.

AFFECTED QUADRATICS.

109. An Affected Quadratic equation is an equation which contains terms of the second degree and also of the first with respect to the unknown quantity. $x^2 - 3x = 12$, $4x + 3ax^2 = \frac{2ax + 3bx^2}{5}$, and $\frac{a^2x^2}{b^2} - 4ax + 3b^2 = 0$ are affected quadratic equations.

110. Prob.—*To solve an Affected Quadratic Equation.*

RULE.—1. REDUCE THE EQUATION TO THE FORM $x^2 + ax = b$, THE CHARACTERISTICS OF WHICH ARE, THAT THE FIRST MEMBER CONSISTS OF TWO TERMS, THE FIRST OF WHICH IS POSITIVE AND SIMPLY THE SQUARE OF THE UNKNOWN QUANTITY, ITS COEFFICIENT BEING UNITY, WHILE THE SECOND HAS THE FIRST POWER OF THE UNKNOWN QUANTITY, WITH ANY COEFFICIENT (a) POSITIVE OR NEGATIVE, INTEGRAL OR FRACTIONAL; AND THE SECOND MEMBER CONSISTS OF KNOWN TERMS (b).

2. ADD THE SQUARE OF HALF THE COEFFICIENT OF THE SECOND TERM TO BOTH MEMBERS OF THE EQUATION.

3. EXTRACT THE SQUARE ROOT OF EACH MEMBER, THUS PRODUCING A SIMPLE EQUATION FROM WHICH THE VALUE OF THE UNKNOWN QUANTITY IS FOUND BY SIMPLE TRANSPOSITION.

DEM.—By definition an affected quadratic equation contains but three kinds of terms, viz: terms containing the square of the unknown quantity, terms containing the first power of the unknown quantity, and *known* terms. Hence each of the three kinds of terms may, by clearing of fractions, transposition, and uniting, as the particular example may require, be united into one, and the results arranged in the order given. If, then, the first term, *i. e.* the one containing the square of the unknown quantity, has a coefficient other than unity, or is negative, its coefficient can be rendered unity or positive without destroying the equation by dividing both the members by whatever coefficient this term may chance to have after the first reductions. The equation will then take the form $x^2 \pm ax = \pm b$. Now adding $\left(\frac{a}{2}\right)^2$ to the first member makes it a perfect square $\left(\text{the square of } x \pm \frac{a}{2}\right)$, since a trinomial is a perfect square when one of its terms (the middle one, ax, in this case) is $\pm$ twice the product of the square roots of the other two, these two being both positive (***116,*** PART I.). But, if we add the square of half the coefficient of the second term to the first member to make it a complete square, we must add it to the second member to preserve the equality of the members. Having extracted the square root of each member, these roots are equal, since like roots of equals are equal. Now, since the first term of the trinomial square is x^2, and the last $\left(\frac{a^2}{4}\right)$ does not contain x, its square root is a binomial consisting of $x \pm$ the square root of its third term, or half the coefficient of the middle term, and hence a known quantity. The square root of the second member can be taken exactly, approximately, or indicated, as the case may be. Finally, as the first term of this resulting equation is simply the unknown quantity, its value is found by transposing the second term.

SCH. 1.—This process of adding the square of half the coefficient of the first power of the unknown quantity to the first member, in order to make it a perfect square, is called COMPLETING THE SQUARE. There are a variety of other ways of completing the square of an affected quadratic, some of which will be given as we proceed; but this is the most important. This method will solve all cases: others are mere matters of convenience, in special cases.

111. COR. 1.—*An affected quadratic equation has two roots. These roots may both be positive, both be negative, or one positive and the other negative. They are both real, or both imaginary.*

DEM.—Let $x^2 + px = q$ be any affected quadratic equation reduced to the form for completing the square. In this form p and q may be either positive or negative, integral or fractional. Solving this equation we have $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$. We will now observe what different forms this expression can take, depending upon the signs and relative values of p and q.

1st. *When* p *and* q *are both positive.* The *signs* will then stand as given; *i.e.*, $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$. Now, it is evident that $\sqrt{\frac{p^2}{4} + q} > \frac{p}{2}$, for $\sqrt{\frac{p^2}{4} + q}$ is the square root of something *more* than $\frac{p^2}{4}$. Hence, as $\frac{p}{2} < \sqrt{\frac{p^2}{4} + q}$, $-\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}$ is *positive;* but $-\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}$ is *negative*, for both parts are negative. Moreover the negative root is numerically greater than the positive, since the former is the numerical sum of the two parts, and the latter the numerical difference. $\therefore$ When p and q are both + in the given form, one root is positive and the other negative, and the negative root is numerically greater than the positive one.

2d. *When* p *is negative and* q *positive.* We then have $x = -\frac{-p}{2} \pm \sqrt{\frac{(-p)^2}{4} + q}$ $= \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$. If we take the plus sign of the radical, x is positive; but if we take the $-$ sign, x is negative, since $\sqrt{\frac{p^2}{4} + q} > \frac{p}{2}$. Moreover, the positive root is numerically the greater. $\therefore$ When p is negative and q positive, one root is positive and the other negative; but the positive root is numerically greater than the negative.

3d. *When* p *and* q *are both negative.* We then have $x = -\frac{-p}{2} \pm \sqrt{\frac{(-p)^2}{4} + (-q)}$ $= \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$. In this if $\frac{p^2}{4} > q$, $\sqrt{\frac{p^2}{4} - q}$ is real, and as it is less than $\frac{p}{2}$, both values are positive. If $\frac{p^2}{4} = q$, $\sqrt{\frac{p^2}{4} - q} = 0$ and there is but one value of x, and this is positive. (It is customary to *call* this *two equal* positive roots for the sake of analogy, and for other reasons which cannot now be appreciated by the pupil.) If $\frac{p^2}{4} < q$, $\sqrt{\frac{p^2}{4} - q}$ becomes the square root of a negative quantity and hence imaginary.

4th. *When* p *is positive and* q *negative.* We then have $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$. As before, this gives two *real* roots when $q < \frac{p^2}{4}$. When this is the case both roots are *negative.* [Let the pupil show how this is seen.] When $q = \frac{p^2}{4}$, the roots are equal and negative; *i. e.*, there is but one. When $\frac{p^2}{4} < q$ both roots are imaginary.

112. Cor. 2.—*An affected quadratic being reduced to the form* $x^2 + px = q$, *the value of* x *can always be written out without taking the intermediate steps of adding the square of half the coefficient of the second term, extracting the root, and transposing. The root in such a case is half the coefficient of the second term taken with the opposite sign,* $\pm$ *the square root of the sum of the square of this half coefficient, and the known term of the equation. This is observed directly from the form* $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$, *and more in detail in the demonstration of the preceding corollary.*

113. Cor. 3.—*Upon the principle that the middle term of a trinomial square is twice the product of the square roots of the other two, we can often complete the square more advantageously than by the regular rule.*

Thus having $4x^2 - 12x = 16$. Since $4x^2$ is a perfect square, and $12x$ is divisible by twice the square root of $4x^2$, *i. e.* by $4x$, we see that the wanting third term is 3^2, or 9. Adding this to both members, we have $4x^2 - 12x + 9 = 25$.

Again, if the coefficient of x^2 is not a perfect square, it can be rendered such by multiplying by itself (or often by some other factor). If then the second term (the term in x) is not divisible by twice the square root of this first term, we may multiply both members of the equation by 4, and the first term will still be a perfect square, and the second term divisible by twice its square root.

114. Sch. 2.—The method of Art. ***110*** is perfectly general, and will solve all cases ; but some may prefer the more elegant methods indicated in (***113***), in special cases. Some illustrations of these methods are given in the examples following.

Examples.

1. $x^2 - 6x = 16$. 2. $3x^2 = 24x - 36$. 3. $x^2 - 4ax = 7a^2$.

4. $x^2 - 7x + 2 = 10$. 5. $3x^2 + 135 = 12x$. 6. $x^2 + (a-1)x = a$.

7. $\frac{x}{x-1} = \frac{3}{2} + \frac{x-1}{2}$. 8. $\frac{4x}{x+7} - \frac{x-7}{2x+3} = 2$. 9. $\frac{a^2x^2}{b^2} - \frac{2ax}{c} + \frac{b^2}{c^2} = 0$.

10. Solve $9x^2 + 12x = 32$, $7x^2 - 14x = -5\frac{5}{7}$, and $3x^2 - 13x = 10$, by Art. ***113***.

Sug's.—Dividing $12x$ by $2\sqrt{9x^2}$, or $6x$, we have 2 as the square root of the third term. Hence $9x^2 + 12x + 4 = 36$, is the equation with the square completed.

$7x^2 - 14x = -5\frac{5}{7}$, becomes, by multiplying by 7, $49x^2 - 98x = -40$. Hence, completing the square as in the last, $49x^2 - 98x + 49 = 9$.

$3x^2 - 13x = 10$, multiplied by 3 and by 4 becomes $36x^2 - 156x = 120$. Hence, completing the square as before, $36x^2 - 156x + (13)^2 = 289$.

[Note.—Solve the following by any of the preceding methods, according to taste or expediency.]

11. $(2x+3)^{\frac{1}{2}} \times (3x+7)^{\frac{1}{2}} = 12$. 12. $3x^2 + 2x = 85$.

13. $a^2(1 + b^2x^2) = b(2a^2x + b)$. 14. $5x^2 - 9x + 2\frac{1}{4} = 0$.

15. $3\sqrt{112 - 8x} = 19 + \sqrt{3x + 7}$. 16. $7x^2 - 11x = 6$.

17. $(x-c)\sqrt{ab}-(a-b)\sqrt{cx}=0.$

18. $3x^2+x=11.$

19. $\frac{5(3x-1)}{1+5\sqrt{x}}+\frac{2}{\sqrt{x}}=3\sqrt{x}.$

20. $\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\frac{x}{a}.$

21. $\frac{\sqrt{1+x}}{1+\sqrt{1+x}}=\frac{\sqrt{1-x}}{1-\sqrt{1-x}}.$

22. $\frac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\frac{5}{11}.$

23. $b\left(\frac{a-b}{x}+1\right)\left(\frac{a-2b}{x}+1\right)=\frac{a^2}{x}-a.$

24. $\frac{90}{x}-\frac{90}{x+1}-\frac{27}{x+2}=0.$

25. $\sqrt{4+\sqrt{2x^3+x^2}}=\frac{x+4}{2}.$

26. $2\sqrt{x}+\frac{2}{\sqrt{x}}=5.$

27. $\frac{x}{3}-4-x^2+2x-\frac{4x^3}{5}=45-3x^2+4x.$

28. $\frac{x+a}{x-2a}+\frac{x-2a}{x+a}=1.$

29. $2\sqrt{x}+\sqrt{4x+\sqrt{7x+2}}=1.$

30. $\frac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}}=(x-2)^2.$

31. $\frac{x^2}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-(a^{\frac{1}{2}}-b^{\frac{1}{2}})x=\frac{1}{(ab^2)^{-\frac{1}{2}}+(a^2b)^{-\frac{1}{2}}}.$

SECTION III.

EQUATIONS OF OTHER DEGREES WHICH MAY BE SOLVED AS QUADRATICS.

115. Prop. 1.—*Any Pure Equation* (i. e., *one containing the unknown quantity affected with but one exponent*) *can be solved in a manner similar to a Pure Quadratic.*

DEM.—In any such equation we can find the value of the unknown quantity affected by its exponent, as if it were a simple equation. If then the unknown quantity is affected with a positive integral exponent it can be freed of it by evolution; if its exponent be a positive fraction it can be freed of it by extracting the root indicated by the numerator of the exponent, and involving this root to the power indicated by the denominator. If the exponent of the unknown quantity is negative it can be rendered positive by multiplying the equation by the unknown quantity with a numerically equal positive exponent. Q. E. D.

116. Prop. 2.—*Any equation containing one unknown quantity affected with only two different exponents, one of which is twice the other, can be solved as an Affected Quadratic.*

DEM.—Let m represent any number, positive or negative, integral or fractional; then the two exponents will be represented by m and $2m$; and the equation can be reduced to the form $x^{2m} + px^m = q$. Now let $y = x^m$, and $y^2 = x^{2m}$, whatever m may be. Substituting we have $y^2 + py = q$, whence $y = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$. But $y = x^m$; hence $x = \left(-\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}\right)^{\frac{1}{m}}$. Q. E. D.

117. Prop. 3.—*Equations may frequently be put in the form of a quadratic by a judicious grouping of terms containing the unknown quantity, so that one group shall be the square root of the other.*

DEM.—This proposition will be established by a few examples, as it is not a general truth, but only points out a special method.

118. COR.—THE FORM OF THE COMPOUND TERM *may sometimes be found by transposing all the terms to the first member, arranging them with reference to the unknown quantity, and extracting the square root. In trying this expedient, if the highest exponent is not even it must be made so by multiplying the equation by the unknown quantity. In like manner the coefficient of this term is to be made a perfect square. When the process of extracting the root terminates, if the root found can be detected as a part, or factor, or factor of a part of the remainder, the root may be the polynomial term.*

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119. Prop. 4.—*When an equation is reduced to the form* $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \cdots + L = 0$, *the roots with their signs changed are factors of the absolute (known) term* L.

DEM.—1st. The equation being in this form, if a is a root, the equation is divisible by $x - a$. For, suppose upon trial $x - a$ goes into the polynomial $x^n + Ax^{n-1} +$, etc., Q times *with a remainder* R. (Q represents any series of terms which may arise from such a division, and R, any remainder.) Now, since the quotient multiplied by the divisor, + the remainder, equals the dividend, we have $(x - a)\,Q + R = x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \cdots + L$. But this polynomial $= 0$. Hence $(x - a)Q + R = 0$. Now, by hypothesis a is a root, and consequently $x - a = 0$. Whence $R = 0$, or there is no remainder.

2d. If now $x - a$ exactly divides $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \cdots + L$, a must exactly divide L, as readily appears from considering the process of division. Hence $-a$ is a factor of L, a being a root of the equation. Q. E. D.

120. Many equations of other degrees than the second, and which do not fall under the preceding cases, may still be solved as quadratics by means of *Special Artifices*. For these artifices the student must depend upon his own ingenuity, after having studied some examples as specimens. These methods are so restricted and special that it is not expedient to classify them; in fact, every expert algebraist is constantly developing new ones. See Ex's. 47–57. The following principle is often of service in such solutions:

121. Prop. 5.—*When an equation can be put in such a form that the product of any number of factors equals* 0, *the equation is satisfied by putting any one of these factors equal to* 0.

DEM.—This scarcely needs demonstration, but will appear evident if we consider such an expression as $(x^2+1)(x^3-x^2+1)(x-1)=0$. Now, on the hypothesis that any factor, as x^2+1, is 0, the equation is satisfied.* So also, if $x^3-x^2+1=0$, the equation is satisfied, etc.

122. SCH.—*Ability to recognize a factor in a polynomial is of prime importance in the solution of such equations. It is the grand key to difficult solutions.*

EXAMPLES.

1. $x^4=81.$ 2. $x^5=32.$ 3. $x^{\frac{2}{3}}=m.$

4. $y^{\frac{5}{3}}=243.$ 5. $z^{\frac{3}{2}}=1331.$ 6. $y^{\frac{1}{3}}=4.$

7. $x^{\frac{m}{n}}=b.$ 8. $x^{\frac{1}{3}}+\sqrt{2}=\dfrac{2}{x^{\frac{1}{3}}-\sqrt{2}}.$ 9. $x^4+4x^2=12.$

10. $x^{2m}+x^m=p.$ 11. $x^3-x^{\frac{3}{2}}=56.$ 12. $ax^{\frac{3}{2}}+bx^{\frac{3}{4}}=c.$

13. $x^n-2ax^{\frac{n}{2}}=b.$ 14. $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756.$ 15. $x^{\frac{1}{4}}+5x^{\frac{1}{2}}-22=0.$

16. $ax^{\frac{2}{3}}-bx^{\frac{1}{3}}-c=0.$ 17. $x^{\frac{1}{3}}+\dfrac{5}{2x^{\frac{1}{3}}}=3\frac{1}{4}.$

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* In strictness we should add "since this hypothesis cannot render any other factor ∞."

18. $3x^n\sqrt[3]{x^n}+\dfrac{2x^n}{\sqrt[3]{x^n}}=16.$ 19. $x^2-2x+6\sqrt{x^2-2x+5}=11.$*

20. $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}.$

21. $x^2-x+5\sqrt{2x^2-5x+6}=\frac{1}{2}(3x+33).$

22. $\sqrt{x+12}+\sqrt[4]{x+12}=6.$ 23. $\sqrt{x-\dfrac{1}{x}}+\sqrt{1-\dfrac{1}{x}}=x.$

24. $1+\sqrt{1-\dfrac{a}{x}}=\sqrt{1+\dfrac{x}{a}}.$ 25. $x^2+\dfrac{1}{x^2}+2\left(x+\dfrac{1}{x}\right)=\dfrac{142}{9}.$

26. $2x^2-2x+2\sqrt{2x^2-7x+6}=5x-6.$

27. $\sqrt[m]{(1+x)^2}-\sqrt[m]{(1-x)^2}=\sqrt[m]{1-x^2}.$†

28. $x^4-8x^3+29x^2-52x+36=126.$

SOLUTION.— See (*118*). Transposing 126, and extracting the square root; when we have the two terms x^2-4x of the root, we have a remainder $13x^2-52x-90$. We now notice that, if we call 4 the next term of the root, the next remainder will be $5x^2-20x-106$, which we may write $5(x^2-4x+4)-126$. Hence our equation may be put in the form $(x^2-4x+4)^2+5(x^2-4x+4)=126$.

29. $x^4-6x^3+5x^2+12x=60.$ 30. $x^3-6x^2+11x=6.$

31. $4x^4+\dfrac{x}{2}=4x^3+33.$ 32. $x^3+5x^2+3x-9=0.$‡

33. $x^3-6x^2+13x-10=0.$ 34. $x^3-13x^2+49x-45=0.$

35. $x^3+8x^2+17x+10=0.$ 36. $x^3-29x^2+198x-360=0.$

37. $x^3-15x^2+74x-120=0.$ 38. $x^4+2x^3-3x^2-4x+4=0.$

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* $x^2-2x+5+6\sqrt{x^2-2x+5}=16.$ Putting $x^2-2x+5=y^2$, $y^2+6y=16$. Such substitution is not absolutely necessary, as we may treat x^2-2x+5 as the unknown quantity without substituting. Solve the following in like manner.

† Dividing by $\sqrt[m]{1-x^2}$, we have $\sqrt[m]{\dfrac{1+x}{1-x}}-\sqrt[m]{\dfrac{1-x}{1+x}}=1$. Then, multiplying by $\sqrt[m]{\dfrac{1+x}{1-x}}$, we have $\sqrt[m]{\left(\dfrac{1+x}{1-x}\right)^2}-1=\sqrt[m]{\dfrac{1+x}{1-x}}$ or $\sqrt[m]{\left(\dfrac{1+x}{1-x}\right)^2}-\sqrt[m]{\dfrac{1+x}{1-x}}=1.$

‡ By (*119*) we are led to try $+1$ or -1, or $+$ or -3, as roots. The equation is divisible by $x-1$, and $x+3$.

39. $x^4-10x^3+35x^2-50x+24=0.$ 40. $x^4-4x^3+8x^2-8x=21.$

41. $x^4-2x^3-25x^2+26x=-120.$ 42. $3x^4+13x^3-117x=243.$

43. $\frac{x}{14}-\frac{30}{7x^3}+\frac{12+\frac{1}{2}x}{3x}=\frac{7}{2x^2}+1\frac{1}{6}.$

44. $x=\frac{12+8\sqrt{x}}{x-5}.$ Put $\sqrt{x}=y.$

Special Expedients.

45. To find the roots of $x^2=\pm 1$, $x^3=\pm 1$, $x^4=\pm 1$, $x^5=\pm 1$, $x^6=\pm 1$, and $x^8=\pm 1$.

Sug's. $x^5-1=0$. Factoring $(x-1)(x^4+x^3+x^2+x+1)=0$. $\therefore x=1$, and also $x^4+x^3+x^2+x+1=0$. Dividing by x^2, $x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0$, or $x^2+2+\frac{1}{x^2}+x+\frac{1}{x}=1$, or $\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=1$.

46. To find the roots of $\frac{1+x^4}{(1+x)^4}=a$.

Sug's. $1+x^4=a(1+x)^4=a(1+4x+6x^2+4x^3+x^4)$. Whence, dividing by x^2 and arranging terms, $x^2+\frac{1}{x^2}-\frac{4a}{1-a}\left(x+\frac{1}{x}\right)=\frac{6a}{1-a}$.

47. To solve $\frac{2a\sqrt{1+x^2}}{1-x+\sqrt{1+x^2}}=a+b$.

Sug's.—This can be cleared of fractions, and then of radicals, in the ordinary way. But the following expedient will be found elegant in this case, and convenient in many. Dividing by $2a$, treating the resulting equation as a proportion, and taking it by division, we have $\frac{1-x}{\sqrt{1+x^2}}=\frac{a-b}{a+b}$. $\therefore \frac{1+x^2-2x}{1+x^2}=\left(\frac{a-b}{a+b}\right)^2$, or $\frac{2x}{1+x^2}=1-\left(\frac{a-b}{a+b}\right)^2=\frac{4ab}{(a+b)^2}$. Taking this again by composition and division, we obtain $\frac{(1+x)^2}{(1-x)^2}=\frac{(a+b)^2+4ab}{(a+b)^2-4ab}=\frac{(a-b)^2+8ab}{(a-b)^2}$, or $\frac{1+x}{1-x}=\frac{\sqrt{(a-b)^2+8ab}}{a-b}$. Again, by division and composition, we obtain

$$x=\frac{\sqrt{(a-b)^2+8ab}-(a-b)}{\sqrt{(a-b)^2+8ab}+(a-b)}.$$

48. To solve $(1+x+x^2)^2 = \frac{a+1}{a-1}(1+x^2+x^4)$.

SUG'S.—Dividing by $1+x+x^2$, $1+x+x^2 = \frac{a+1}{a-1}(1-x+x^2)$, or $\frac{1+x+x^2}{1-x+x^2} = \frac{a+1}{a-1}$. $\therefore \frac{1+x^2}{x} = a$.

49. To solve $a = x^4 + (1-x)^4$.

SUG'S.—Since $(1-x)^4 = (x-1)^4$, we may write $a = (x-\frac{1}{2}+\frac{1}{2})^4 + (x-\frac{1}{2}-\frac{1}{2})^4$. Now put $x-\frac{1}{2} = y$, substitute and expand.

50. To solve $\sqrt{x-\frac{1}{x}} - \sqrt{1-\frac{1}{x}} = \frac{x-1}{x}$.

SUG'S.—Dividing by $\sqrt{1-\frac{1}{x}}$, $\sqrt{x+1}-1 = \frac{\sqrt{x-1}}{\sqrt{x}}$. Squaring, etc., $2\sqrt{x+1} = 1+\frac{1}{x}+x$. Squaring, etc., again, $\left(x-\frac{1}{x}\right)^2 - 2\left(x-\frac{1}{x}\right) = -1$.

51. Solve $x^2 - x + 3\sqrt{2x^2-3x+2} = \frac{x}{2} + 7$.

52. Solve $\frac{9}{1+x+x^2} = 5 - x - x^2$.

53. Solve $\frac{a^2+ax+x^2}{a^2-ax+x^2} = \frac{a^2}{x^2}$.

54. Solve $\frac{x-\sqrt{x^2-a^2}}{\sqrt{x+\sqrt{x^2-a^2}}} = \sqrt[4]{x^2-a^2}\,(\sqrt{x^2+ax}-\sqrt{x^2-ax})$.

55. Solve $2x\sqrt{1-x^4} = a(1+x^4)$. Also $\frac{1+x^3}{(1+x)^3} = \frac{13}{25}$.

56. Solve $6x^3 - 5x^2 + x = 0$. Also $x^3 + x^2 - 4x - 4 = 0$.

57. Solve $8x^3 + 16x = 9$. Also $3x^6 + 8x^4 - 8x^2 = 3$.

SUG.—The solutions of the last four depend upon the recognition of a common factor.

SECTION IV.

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE BETWEEN TWO UNKNOWN QUANTITIES.

123. Prop. 1.—*Two equations between two unknown quantities, one of the second degree and the other of the first, may always be solved as a quadratic.*

DEM.—The general form of a *Quadratic Equation* between two unknown quantities is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

since in every such equation all the terms in x^2 can be collected into one, and its coefficient represented by a; all those in xy can also be collected into one, and its coefficient represented by b, etc., etc.

The general form of an equation of the *First Degree* between two variables is

$$a'x + b'y + c' = 0.$$

Now, from the latter $x = \frac{-b'y - c'}{a'}$. which substituted in the former gives no term containing a higher power of y than the second, and hence the resulting equation is a quadratic. Q. E. D.

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124. Prop. 2.—*In general, the solution of two quadratics between two unknown quantities, requires the solution of a biquadratic; that is, an equation of the fourth degree.*

DEM.—Two General Equations between two unknown quantities have the forms

$$(1) \qquad ax^2 + bxy + cy^2 + dx + ey + f = 0, \text{ and}$$

$$(2) \qquad a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0.$$

$$\text{From (1), } x = -\frac{by + d}{2a} \pm \sqrt{\frac{(by + d)^2}{4a^2} - \frac{cy^2 + ey + f}{a}}.$$

Now, to substitute this value of x in equation (2), it must be squared, and also, in another term, multiplied by y, either of which operations produces rational terms containing y^2, and a radical of the second degree. Then, to free the resulting equation of radicals will require the squaring of terms containing y^2, which will give terms in y^4, as well as other terms. Q. E. D.

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125. DEF.—***A Homogeneous Equation*** is one in which each term contains the same number of factors of the unknown quantities. $2x^2 - 3xy - y^2 = 16$ is homogeneous. $3x^2 - 2y + y^2 = 10$ is not homogeneous.

126. Prop. 3.—*Two Homogeneous Quadratic Equations between two unknown quantities can always be solved by the method of quadratics, by substituting for one of the unknown quantities the product of a new unknown quantity into the other.*

DEM.—The truth of this proposition will be more readily apprehended by means of a particular example. Take the two homogeneous equations $x^2 - xy + y^2 = 21$, and $y^2 - 2xy + 15 = 0$. Let $x = vy$, v being a new unknown quantity, called an auxiliary, whose value is to be determined. Substituting in the given equations, we have $v^2y^2 - vy^2 + y^2 = 21$, and $y^2 - 2vy^2 = -15$. From these we find $y^2 = \frac{21}{v^2 - v + 1}$, and $y^2 = \frac{15}{2v - 1}$. Equating these values of y^2, $\frac{21}{v^2 - v + 1} = \frac{15}{2v - 1}$; whence $42v - 21 = 15v^2 - 15v + 15$. This latter equation is an affected quadratic, which solved for v, gives $v = 3$, and $\frac{4}{5}$. Knowing the values of v we readily determine those of y from $y^2 = \frac{15}{2v - 1}$, and find $y = \pm \sqrt{3}$ when $v = 3$, and $y = \pm 5$ when $v = \frac{4}{5}$. Finally as $x = vy$, its values are $x = \pm 3\sqrt{3}$, and ± 4.

By observing the substitution of vy for x in this solution, it is seen that it brings the square of y in every term containing the unknown quantities, in each equation, and hence enables us to find two values of y^2 in terms of v. It is easy to see that this will be the case in any homogeneous quadratic with two unknown quantities, for we have in fact, in the first of the given equations, all the variety of terms which such an equation can contain. Again, that the equation in v will not be higher than the second degree is evident, since the values of y^2 consist of known quantities for numerators, and can have denominators of only the second, or second and first degrees with reference to v. Whence v can always be determined by the method of quadratics; and being determined, the value of y is obtained from a *pure* quadratic ($y^2 = \frac{15}{2v - 1}$, in this case), and that of x from a *simple* equation ($x = vy$ in this case).

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127. Prop. 4.—*When the unknown quantities are similarly involved in two quadratic, or even higher equations, the solution can often be effected as a quadratic, by substituting for one of the unknown quantities the sum of two others, and for the other unknown quantity the difference of these new quantities.*

As this is only a *special* expedient, and not a *general* principle, its truth will be rendered sufficiently evident by the solution of a few examples. See Ex's. 13, 14, 15.

EXAMPLES.

1. $\begin{cases} 7x^2-8xy=159, \\ 5x + 2y=7. \end{cases}$

2. $\begin{cases} x^2-2xy-y^2=1, \\ x+y=2. \end{cases}$

3. $\begin{cases} x+y=4, \\ \frac{1}{x}+\frac{1}{y}=1. \end{cases}$

4.* $\begin{cases} x^2+y^2=65, \\ xy=28. \end{cases}$

5. $\begin{cases} x^2+xy=15, \\ xy-y^2=2. \end{cases}$

6. $\begin{cases} x^2+xy+4y^2=6, \\ 3x^2+8y^2=14. \end{cases}$

7. $\begin{cases} x^2+xy+2y^2=74, \\ 2x^2+2xy+y^2=73. \end{cases}$

8. $\begin{cases} x^2+xy=12, \\ xy+y^2=2. \end{cases}$

9. $\begin{cases} x^2-4y^2=9, \\ xy+2y^2=3. \end{cases}$

10. $\begin{cases} x^2+y^2+1=3xy, \\ 2(xy+4)=3y^2. \end{cases}$

11. $\begin{cases} x^2+xy+y^2=52, \\ xy-x^2=8. \end{cases}$

12. $\begin{cases} x^2-2xy-y^2=31, \\ \frac{1}{2}x^2+2xy-y^2=101. \end{cases}$

13. $\begin{cases} 4(x+y)=3xy, \\ x+y+x^2+y^2=26. \end{cases}$

14. $\begin{cases} x^2+y^2=\frac{5}{2}xy, \\ x-y=\frac{1}{4}xy. \end{cases}$

15. $\begin{cases} xy(x+y)=30, \\ x^3+y^3=35. \end{cases}$

SUG.—The last three are readily solved by (*127*). Thus, in the 15th, putting $x=z+v$, and $y=z-v$, the equations become $2z^3-2v^2z=30$, and $2z^3+6v^2z=35$.

16. $\begin{cases} \frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{5}{2}, \\ x^2+y^2=20. \end{cases}$

17. $\begin{cases} x+y-\sqrt{xy}=7, \\ x^2+y^2+xy=133. \end{cases}$

18. $\begin{cases} x-y=8, \\ x^4-y^4=14560. \end{cases}$

SPECIAL SOLUTIONS.

19. $y^2-4xy+20x^2+3y-264x=0,\quad 5y^2-38xy+x^2-12y+1056x=0.$

SUG.—Add 4 times the first to the second.

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* Two *homogeneous* quadratics can always be solved by (*126*), but special expedients are often more elegant. In this case by adding twice the second to the first, and extracting the square root, we have $x+y=\pm 11$. Subtracting twice the second from the first, and extracting the square root, we have $x-y=\pm 3$.

20. $x + y = x^2, \qquad 3y - x = y^2.$

SUG.—Subtract the first from the second.

21. $\begin{cases} x - y = 2, \\ x^3 - y^3 = 8. \end{cases}$ 22. $\begin{cases} x + y = 5, \\ x^3 + y^3 = 65. \end{cases}$ 23. $\begin{cases} x + y = 4, \\ x^4 + y^4 = 82. \end{cases}$

SUG'S.—To solve the 23d, square the first, writing the result $x^2 + y^2 = 16 - 2xy$, and square again. Then for $x^4 + y^4$ substitute 82.

24. To solve $x - y = 3$, and $x^5 - y^5 = 3093$.

SUG'S.—Divide the second by the first, and proceed in a manner similar to that given for the last.

25. To solve $x^2 - xy + y^2 = 7$, and $x^4 + x^2y^2 + y^4 = 133$.

SUG.—Divide the second by the first.

26. To solve $\left(3 - \frac{6y}{x+y}\right)^2 + \left(3 + \frac{6y}{x-y}\right)^2 = 82$, and $xy = 2$.

SUG'S.—Write the first $\left(\frac{3x-3y}{x+y}\right)^2 + \left(\frac{3x+3y}{x-y}\right)^2 = 82$; and put $\frac{x-y}{x+y} = v$. Whence $9v^2 + \frac{9}{v^2} = 82$.

27. To solve $x^3 + y(xy - 1) = 0$, and $y^3 - x(xy + 1) = 0$.

SUG'S.—Write $x^4 + x^2y^2 - xy = 0$, and $y^4 - x^2y^2 - xy = 0$, and subtract the second from the first. Whence $x^4 - y^4 + 2x^2y^2 = 0$, or $x^4 + 2x^2y^2 + y^4 = 2y^4$, and $x^2 + y^2 = \sqrt{2}\, y^2$, or $\frac{x}{y} = \sqrt{\sqrt{2} - 1}$. From the given equations we get $\frac{1 - xy}{1 + xy} = \frac{x^4}{y^4}$. Hence $\frac{1 - xy}{1 + xy} = 3 - 2\sqrt{2}$, or $xy = \frac{1}{2}\sqrt{2}$.

28. Given $xy = a(x + y)$, $xz = b(x + z)$, and $yz = c(y + z)$.

SUG.—These are readily put into the forms $\frac{1}{a} = \frac{1}{y} + \frac{1}{x}$, $\frac{1}{b} = \frac{1}{z} + \frac{1}{x}$, and $\frac{1}{c} = \frac{1}{z} + \frac{1}{y}$.

29. Given $x(x+y+z) = 18$, $y(x+y+z) = 12$, and $z(x+y+z) = 6$.

30. Given $xyz = 48$, $\frac{x}{yz} = \frac{1}{12}$, and $\frac{xy}{z} = \frac{4}{3}$.

31. Given $x + y + z = 6$, $4x + y = 2z$, and $x^2 + y^2 + z^2 = 14$.

32. Given $2\sqrt{x^2 - y^2} + xy = 26$, and $\frac{x}{y} - \frac{y}{x} = \frac{9}{20}$.

33. Given $\dfrac{x+y}{x-y} + 10\dfrac{x-y}{x+y} = 7$, and $xy^3 = 3$.

34. Given $y(x^2 + y^2) = 4(x+y)^2$, and $xy = 4(x+y)$.

35. Given $x + y = 10$, and $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = \dfrac{5}{2}$.

36. Given $\sqrt{x} - \sqrt{y} = 2\sqrt{xy}$, and $x + y = 20$.

37. Given $\sqrt{x^2 + y^2} + \sqrt{x^2 - y^2} = 2y$, and $x^4 - y^4 = a^4$.

38. Given $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = \dfrac{7}{\sqrt{xy}} + 1$, and $\sqrt{x^3y} + \sqrt{xy^3} = 78$.

39. Given $\sqrt{x+y} + 2\sqrt{x-y} = \dfrac{2(x-1)}{\sqrt{x-y}}$, and $\dfrac{x^2 + y^2}{xy} = \dfrac{34}{15}$.

40. $\begin{cases} y^2 - 64 = 8x^{\frac{1}{2}}y, \\ y - 4 = 2y^{\frac{1}{2}}x^{\frac{1}{2}}. \end{cases}$ 41. $\begin{cases} x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3x, \\ x^{\frac{1}{2}} + y^{\frac{1}{3}} = x. \end{cases}$ 42. $\begin{cases} 8x^{\frac{1}{3}} - y^{\frac{1}{2}} = 14, \\ x^{\frac{2}{3}}y^{\frac{3}{2}} = 2y^2. \end{cases}$

Applications.

1. The plate of a looking-glass is 18 inches by 12, and it is to be surrounded by a plain frame of uniform width, and of surface equal to that of the glass. Required the width of the frame.

2. A person bought some fine sheep for \$360, and found that if he had bought 6 more for the same money, he would have paid \$5 less for each. How many did he buy, and what was the price of each?

3. A traveller sets out for a certain place, and travels one mile the first day, two the second, three the third, and so on: in 5 days afterward another sets out, and travels 12 miles a day. How long and how far must he travel before they will come together?

4. Divide the number 48 into two such parts that their product may be 432.

5. Divide the number 24 into two such parts that their product may be equal to 35 times their difference.

6. For a journey of 108 miles, 6 hours less would have sufficed, had the traveller gone 3 miles an hour faster. At what rate did he travel?

7. The fore wheel of a coach makes 6 revolutions more than the hind wheel in going 120 yards; but, if the circumference of each wheel be increased by 1 yard, the fore wheel will make only 4 revolutions more than the hind wheel in the 120 yards. What is the circumference of each wheel?

8. The product of two numbers is p; and the difference of their cubes is equal to m times the cube of their difference. Find the numbers.

9. Find two numbers whose product is equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

10. There are 4 numbers in arithmetical progression. The sum of the extremes is 8; and the product of the means is 15. What are the numbers?

SUG.—In solving examples involving several quantities in arithmetical progression, it is usually expedient to represent the middle one of the series, when the number of terms is *odd*, by x, and let y be the common difference. If the number of terms is *even*, represent the *two middle terms* by $x - y$, and $x + y$, making the common difference $2y$.

11. Five persons undertake to reap a field of 87 acres. The five terms of an arithmetical progression, whose sum is 20, will express the times in which they can severally reap an acre, and they all together can finish the job in 60 days. In how many days can each, separately, reap an acre?

12. There are three numbers in geometrical progression, the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

SUG'S.—In solving examples involving several quantities in geometrical progression, it is sometimes expedient to represent the first by x, and the ratio by y, so that the numbers will be x, xy, xy^2, etc. In other cases it is expedient, if the number of numbers sought is odd, to make xy the middle term of the series and $\frac{y}{x}$ the ratio. Thus 5 terms will be represented $\frac{x^3}{y}$, x^2, xy, y^2, $\frac{y^3}{x}$. When the number of numbers sought is even, it is sometimes expedient to represent the two means by x and y, and the ratio by $\frac{y}{x}$. Thus 4 terms become $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$.

13. There are three numbers in geometrical progression whose continued product is 64, and the sum of their cubes is 584. Required the numbers.

14. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. Required the numbers.

15. There are three numbers in geometrical progression, whose product is 64, and sum 14. What are the numbers?

16. It is required to find four numbers in arithmetical progression, such that if they are increased by 2, 4, 8, and 15 respectively, the sums shall be in geometrical progression.

17. It is required to find four numbers in geometrical progression such, that their sum shall be 15, and the sum of their squares 85.

18. The sum of 700 dollars was divided among four persons, A, B, C, and D, whose shares were in geometrical progression; and the difference between the greatest and least, was to the difference between the two means, as 37 to 12. What were the several shares?

19. The sum of three numbers in harmonical proportion is 191, and the product of the first and third is 4032; required the numbers.

20. The 2d and 6th terms of a geometrical progression are respectively 21 and 1701. What is the first term, and what the ratio?

21. A and B travel on the same road, at the same rate, and in the same direction. When A is 50 miles from the town D, he overtakes another traveller who goes at the rate of 3 miles in 2 hours; and two hours after, he meets a second traveller who goes at the rate of 9 miles in 4 hours. B overtakes the first traveller 45 miles from D, and meets the second 40 minutes before he (B) reaches the 31st milestone from D. How far are A and B apart?

22. The joint stock of two partners, A and B, was \$2080. A's money was in trade 9 months, and B's 6 months, when they shared stock and gain, A receiving \$1140 and B \$1260. What was each man's stock?

23. There is a number consisting of three digits, the first of which is to the second as the second is to the third; the number itself is to the sum of its digits as 124 to 7; and if 594 be added to it the digits will be inverted. What is the number?

24. A person has \$1300, which he divides into two portions, and loans at different rates of interest, so that the two portions produce

equal returns. If the first portion had been loaned at the second rate of interest, it would have produced $36, and if the second portion had been loaned at the first rate of interest, it would have produced $49. Required the rates of interest.

25. A person traveling from a certain place, goes 1 mile the first day, 2 the second, 3 the third, and so on; and in six days after, another sets out from the same place to overtake him, and travels uniformly 15 miles a day. How many days must elapse after the second starts before they come together?

CHAPTER IV.

INEQUALITIES.

128. An Inequality is an expression in mathematical symbols, of inequality between two numbers or sets of numbers.

ILL.—Thus $a > b$ (read "a greater than b") is an inequality; also $a^2x - 3 < 5 + 2$ (read "$a^2x - 3$ less than $5 + 2$"). (See PART I., ***43.***)

129. Fundamental Principle.—In comparing two positive numbers, that is called the greater which is numerically so. Thus $5 > 3$. But, in comparing two negative numbers, that is called the greater which is numerically the less. Thus $-5 < -3$. Of course *any negative* number is less than *any positive number*. In general, we call $a > b$ when $a - b$ is positive, and $a < b$ when $a - b$ is negative.

130. The part of an inequality at the left of the sign $>$, or $<$, is called the *first member*, and the part at the right, the *second member* of the inequality.

131. For the purposes of mathematical investigation, inequalities are subjected to the same transformations as equations, but with certain characteristic differences in the results, which will be pointed out in the following propositions.

132. If, in transforming an inequality, the same member that was the greater before the transformation is the greater after, the inequality is said to continue to exist *in the same sense;* but, if the transformation changes the general relation of the members, so that the member which was the *greater before* the transformation is the *less after*, the inequality is said to exist *in an opposite sense* in the two inequalities.

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133. Prop.—*The sense in which an inequality exists is not changed,*

1st. *By adding equals to both members, or subtracting equals from both;*

2d. *By multiplying or dividing the members by equal positive numbers;*

3d. *By adding or multiplying the corresponding members of two inequalities which exist in the same sense, if all the members are essentially positive;*

4th. *By raising both members to any power whose index is an odd number;*

5th. *By raising both members to any power, if both members are essentially positive;*

6th. *By extracting the same root of both members, if when the degree of the root is even, only the positive roots be compared.*

ILL. and DEM.—The 1st is, in general, an axiom. Thus if $a > b$, it is evident that $a \pm c > b \pm c$. When $c > a$, $a - c$ is negative, but since $b < a$, $b - c$ is also negative and *numerically* greater than $a - c$. Therefore, in this case, $a - c > b - c$ (**129**).

2d. This is wholly axiomatic. If $a > b$ it is evident that $ma > mb$, and that $\frac{a}{m} > \frac{b}{m}$.

3d. This, too, is an axiom. If $a > b$, and $c > d$, a, b, c, and d being each +, it is evident that $a + c > b + d$; and that $ac > bd$.

4th. This becomes evident by considering that if $a > b$, raising both members to any power whose degree is *odd* will leave the *signs* of the members as at the first, and also the sense of the *numerical* inequality the same.

5th. This appears from the fact that neither the signs nor the sense of the numerical inequality of the members is changed by the process.

6th. This is evident from the fact that the greater number has the greater root, if only positive roots are considered.

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134. Prop.—*The sense in which an inequality exists is changed,*

1st. *By changing the signs of both members;*

2d. *By multiplying or dividing both members by the same negative quantity;*

3d. *By raising both members to the same even power, if the members are both negative in the first instance;*

4th. *By comparing the negative even roots (the members, in the first instance, being both essentially positive).*

ILL. and DEM.—The first is evident, since if $a > b$, $-a < -b$ by (**129**). That is, of two *negative* quantities the numerically greater is really the less.

2d. These operations do not change the *numerical* relation of the members, but do change the signs of the members; hence it falls under the preceding.

3d. and 4th. Essentially the same reasoning as in the last.

EXERCISES.

1. When a and b are unequal, show that $a^2 + b^2 > 2ab$.

SOLUTION.—Let $a > b$; whence $a-b>0$, or $a^2-2ab+b^2>0$, or $a^2+b^2>2ab$. Similarly if $a < b$.

2. Prove that the arithmetical mean between two quantities is, in general, greater than the geometrical. How if the quantities are equal?

3. If a, b, c, are such that the sum of any two is greater than the third, show that $a^2 + b^2 + c^2 < 2(ab + ac + bc)$.

4. If $a^2 + b^2 + c^2 = 1$, and $m^2 + n^2 + r^2 = 1$, show that $am + bn + cr < 1$. How if $a = b = c = m = n = r$?

5. Show that, in general, $(a+b-c)^2 + (a+c-b)^2 + (b+c-a)^2 > ab+bc+ac$. How if $a=b=c$?

6. Which is greater, $2x^3$ or $x + 1$?

SOLUTION.—1st. If $x > 1$, $x^2 > 1$ (?), $2x^3 > 2x$ (?); but $2x > x + 1$ (?). $\therefore 2x^3 > x + 1$.

If $x < 1$, a similar process shows $2x^3 < x + 1$.

7. From $5x - 6 < 3x + 8$, and $2x + 1 < 3x - 3$, show that x may have any value between 7 and 4; *i. e.*, that the limiting values are 7 and 4.

8. What are the limiting values of x determined from the conditions $3x - 2 > \frac{1}{2}x - \frac{4}{5}$, and $\frac{7}{6} - \frac{3}{4}x < 8 - 2x$?

9. The double of a number diminished by 5 is greater than 25, and triple of the number diminished by 7 is less than the double increased by 13. What numbers will satisfy the conditions?

PART III.

AN ADVANCED COURSE IN ALGEBRA.

CHAPTER I.

INFINITESIMAL ANALYSIS.

SECTION I.

DIFFERENTIATION.

135. In certain classes of problems and discussions the quantities involved are distinguished as *Constant* and *Variable*.

136. A Constant quantity is one which maintains the same value throughout the same discussion, and is represented in the notation by one of the leading letters of the alphabet.

137. Variable quantities are such as may assume in the same discussion any value within certain limits determined by the nature of the problem, and are represented by the final letters of the alphabet.

ILL.—If x is the radius of a circle and y is its area, $y = \pi x^2$, as we learn from Geometry, π being about 3.1416. Now if x, the radius, varies, y, the area, will vary; but π remains the same for all values of x and y. In this case x and y are the variables, and π is a constant.

Again, if y is the distance a body falls in time x, it is evident that the greater x is, the greater is y, *i. e.*, that as x varies y varies. We learn from Physics that $y = 16\frac{1}{12}x^2$, for comparatively small distances above the surface of the earth. In the expression $y = 16\frac{1}{12}x^2$, x and y are the variables, and $16\frac{1}{12}$ is a constant.

Once more, suppose we have $y^2 = 25x^3 - 3x^2 - 5$, as an expressed relation between x and y, and that this is the *only* relation which is required to exist between them; it is evident that we may give values to x *at pleasure*, and thus obtain corresponding values for y. Thus if $x=1$, $y=\pm\sqrt{17}$, if $x=2$, $y=\pm\sqrt{183}$, etc., etc. In such a case x and y are called variables. But we notice that if we give to x such a value as to make $3x^2+5>25x^3$ (as, for example, $\frac{1}{2}$, $\frac{1}{4}$, etc.), y will be imaginary. This is the kind of limitation referred to in our definition of variables.*

138. SCH.—The pupil needs to guard against the notion that the terms *constant* and *variable* are synonyms for *known* and *unknown*, and the more so as the notation might lead him into this error. The quantities he has been accustomed to consider in Arithmetic and Elementary Algebra have *all* been constant. The distinction here made is a new one to him, and pertains to a new class of problems and discussions.

139. A Function is a quantity, or a mathematical expression, conceived as depending for its value upon some other quantity or quantities.

ILL.—A man's wages *for a given time* is a function of the amount received per day, or, in general, his wages is a function of the time he works and the amount he receives per day. In the expression $y=16\frac{1}{12}x^2$ (***137***), second illustration, y is a function of x, *i. e.*, the space fallen through is a function of the time. The expression $2ax^2-3x\div 5b$, or any expression containing x, may be spoken of as a function of x.

140. When we wish to indicate that one variable, as y, is a function of another, as x, and do not care to be more specific, we write $y=f(x)$, and read "y equals (or is) a function of x." This means nothing more than that y is equal to some expression containing the variable x, and which may contain any constants. If we wish to indicate several *different* expressions each of which contains x, we write $f(x)$, $\varphi(x)$, or $f'(x)$, etc., and read "the f function of x," "the φ function of x," or "the f' function of x."

ILL.—The expression $f(x)$ may stand for x^3-2x+5, or for $3(a^2-x^2)$, or for *any* expression containing x combined in any way with itself or with constants. But in the same discussion $f(x)$ will mean the same thing throughout. So again, if in a particular discussion we have a certain expression containing x (*e. g.*, $3x^2-ax+2ab$), it may be represented by $f(x)$, while some *other* function of x (*e. g.*, $5(a^3-x^3)+2x^2$) might be represented by $f'(x)$, or $\varphi(x)$.

141. In equations expressing the relation between two variables, as in $y^2=3ax^3-x^2$, it is customary to speak of one of the variables, as y, as a function of the other x. Moreover, it is convenient to think

* The limits of this volume do not permit the interpretation of imaginaries as other than impossible quantities, *i. e.*, inconsistent with the restricted view taken of the particular problem which may be under consideration.

of x as varying and thus producing change in y. When so considered, x is called the *Independent* and y the *Dependent* variable. Or we may speak of y as a function of the variable x.

142. *An Infinitesimal* is a quantity conceived under such a form, or law, as to be necessarily less than any assignable quantity.

Infinitesimals are the increments by which continuous number, or quantity (*8*), may be *conceived* to change value, or grow.

Ill.—*Time* affords a good illustration of continuous quantity, or number. Thus a period of time, as 5 hours, increases, or grows, to another period, as 7 hours, by infinitesimal increments, *i. e.*, not by hours, minutes, or even seconds, but by elements which are less than *any* assignable quantity. In this way we may conceive any continuous, variable quantity to change value, or grow, by infinitesimal increments.

143. *Consecutive Values* of a function, or variable, are values which differ from each other by less than any assignable quantity, *i. e.*, by an infinitesimal part of either.

144. *A Differential* of a function, or variable, is the difference between two consecutive states of the function, or variable. It is the same as an infinitesimal.

Ill.—Resuming the illustration $y = 16\frac{1}{12}x^2$ (*137*), let x be thought of as some particular period of time (as 5 seconds), and y as the distance through which the body falls in that time. Also, let x' represent a period of time infinitesimally greater than x, and y' the distance through which the body falls in time x'. Then x and x' are consecutive values of x, and y and y' are consecutive values of y. Again, the difference between x and x', as $x' - x$, is a differential of the variable x, and $y' - y$ is a differential of the function y.

145. *Notation.*—A differential of x is expressed by writing the letter d before x, thus dx. Also, dy means, and is read "differential y."

Caution.—Do not read dx by naming the letters as you do ax; but read it "differential x." The d is not a factor, but an abbreviation for the word *differential.*

146. *To Differentiate* a function is to find an expression for the increment of the function due to an infinitesimal increment of the variable; or it is the process of finding the relation between the infinitesimal increment of the variable and the corresponding increment of the function.

Rules for Differentiating.

147. *RULE* 1.—To differentiate a single variable, simply write the letter d before it.

This is merely doing what the notation requires. Thus if x and x' are consecutive states of the variable x, *i. e.*, if x' is what x becomes when it has taken an infinitesimal increment, $x' - x$ is the differential of x, and is to be written dx. In like manner, $y' - y$ is to be written dy, y' and y being consecutive values.

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148. *RULE* 2.—Constant factors or divisors appear in the differential the same as in the function.

Dem.—Let us take the function $y = ax$, in which a is any constant, integral or fractional. Let x take an infinitesimal increment dx, becoming $x + dx$; and let dy be the corresponding* increment of y, so that when x becomes $x + dx$, y becomes $y + dy$. We then have

1st state of the function - - - - - - - - - - $y = ax$;

2d, or consecutive state - - - - - - - - $y + dy = a(x + dx) = ax + adx.$

Subtracting the 1st from the 2d - - - - - - $dy = adx$,

which result being the difference between two consecutive states of the function, is its differential (**144**). Now a appears in the differential just as it was in the function. This would evidently be the same if a were a fraction, as $\frac{1}{m}$. We should then have, in like manner, $dy = \frac{1}{m}dx$ as the differential of $y = \frac{1}{m}x$. Q. E. D.

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149. *RULE* 3.—Constant terms disappear in differentiating; or the differential of a constant is 0.

Dem.—Let us take the function $y = ax + b$, in which a and b are constant. Let x take an infinitesimal increment and become $x + dx$; and let dy be the increment which y takes in consequence of this change in x, so that when x becomes $x + dx$, y becomes $y + dy$. We then have

1st state of the function - - - - - - - - $y = ax + b$;

2d, or consecutive state - - - - $y + dy = a(x + dx) + b = ax + adx + b$

Subtracting the 1st from the 2d - - - - $dy = adx$,

which being the difference between two consecutive states of the function, is its differential (**144**). Now from this differential the constant b has disappeared.

We may also say that as a constant retains the same value, there is no differ-

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* The word "contemporaneous" is often used in this connection.

ence between its consecutive states (properly it has no consecutive states). Hence the differential of a constant may be spoken of (though with some latitude) as 0. Q. E. D.

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150. *RULE* 4.—TO DIFFERENTIATE THE ALGEBRAIC SUM OF SEVERAL VARIABLES, DIFFERENTIATE EACH TERM SEPARATELY AND CONNECT THE DIFFERENTIALS WITH THE SAME SIGNS AS THE TERMS.

DEM.—Let $u = x + y - z$, u representing the algebraic sum of the variables x, y, and $-z$. Then is $du = dx + dy - dz$. For let dx, dy, and dz be infinitesimal increments of x, y, and z; and let du be the increment which u takes in consequence of the infinitesimal changes in x, y, and z. We then have

1st state of the function - - - - - - - - - $u = x + y - z$;

2d, or consecutive state - - - - - $u + du = x + dx + y + dy - (z + dz)$,

Or - - - - - - - - - - - - - - - $u + du = x + dx + y + dy - z - dz$.

Subtracting the 1st state from the 2d - - - $du = dx + dy - dz$. Q. E. D.

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151. *RULE* 5.—THE DIFFERENTIAL OF THE PRODUCT OF TWO VARIABLES IS THE DIFFERENTIAL OF THE FIRST INTO THE SECOND, PLUS THE DIFFERENTIAL OF THE SECOND INTO THE FIRST.

DEM.—Let $u = xy$ be the first state of the function. The consecutive state is $u + du = (x + dx)(y + dy) = xy + ydx + xdy + dx \cdot dy$. Subtracting the 1st state from the consecutive state we have the differential, *i. e.*, $du = ydx + xdy + dx \cdot dy$. But, as $dx \cdot dy$ is the product of two infinitesimals, it is infinitely less than the other terms (ydx and xdy), and hence, having no value as compared with them, is to be dropped.* Therefore $du = ydx + xdy$. Q. E. D.

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152. *RULE* 6.—THE DIFFERENTIAL OF THE PRODUCT OF SEVERAL VARIABLES IS THE SUM OF THE PRODUCTS OF THE DIFFERENTIAL OF EACH INTO THE PRODUCT OF ALL THE OTHERS.

DEM.—Let $u = xyz$; then $du = yzdx + xzdy + xydz$. For the 1st state of the function is $u = xyz$, and the 2d, or consecutive state, $u + du = (x + dx)(y + dy)(z + dz)$, or $u + du = xyz + yzdx + xzdy + xydz + xdydz + ydxdz + zdxdy$

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* It will doubtless appear to the pupil, at first, as if this gave a result only *approximately* correct. Such is not the fact. The result is *absolutely* correct. *No* error is introduced by dropping $dx \cdot dy$. In fact this term *must* be dropped according to the nature of infinitesimals. Notice that by definition a quantity which is infinitesimal with respect to another is one which has *no* assignable magnitude with reference to that other. Hence we must so treat it in our reasoning. Now $dx \cdot dy$ is an infinitesimal of an infinitesimal (*i. e.*, two infinitesimals multiplied together), and hence is infinitesimal with reference to ydx and xdy, and must be treated as having *no* assignable value with respect to them; that is, it must be dropped.

$+ dxdydz$. Subtracting, and dropping all infinitesimals of infinitesimals (see preceding rule and foot-note), we have $du = yzdx + xzdy + xydz$.

In a similar manner the rule can be demonstrated for any number of variables. Q. E. D.

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153. *RULE* 7.—THE DIFFERENTIAL OF A FRACTION HAVING A VARIABLE NUMERATOR AND DENOMINATOR IS THE DIFFERENTIAL OF THE NUMERATOR MULTIPLIED BY THE DENOMINATOR, MINUS THE DIFFERENTIAL OF THE DENOMINATOR MULTIPLIED BY THE NUMERATOR, DIVIDED BY THE SQUARE OF THE DENOMINATOR.

DEM.—Let $u = \frac{x}{y}$; then is $du = \frac{ydx - xdy}{y^2}$. For, clearing of fractions, $yu = x$. Differentiating this by Rule 5th, we have $udy + ydu = dx$. Substituting for u its value $\frac{x}{y}$, this becomes $\frac{xdy}{y} + ydu = dx$. Finding the value of du, we have $du = \frac{ydx - xdy}{y^2}$. Q. E. D.

154. COR.—*The differential of a fraction having a constant numerator and a variable denominator is the product of the numerator with its sign changed into the differential of the denominator, divided by the square of the denominator.*

Let $u = \frac{a}{y}$. Differentiating this by the rule and calling the differential of the constant (a) 0, we have $du = \frac{0 - ady}{y^2} = \frac{-ady}{y^2}$. Q. E. D.

155. SCH.—If the numerator is variable and the denominator constant, it falls under Rule 2.

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156. *RULE* 8.—THE DIFFERENTIAL OF A VARIABLE AFFECTED WITH AN EXPONENT IS THE CONTINUED PRODUCT OF THE EXPONENT, THE VARIABLE WITH ITS EXPONENT DIMINISHED BY 1, AND THE DIFFERENTIAL OF THE VARIABLE.

DEM.—1st. *When the exponent is a positive integer.* Let $y = x^m$, m being a positive integer; then $dy = mx^{m-1}dx$. For $y = x^m = x \cdot x \cdot x \cdot x \cdot$ to m factors. Now, differentiating this by Rule 6, we have $dy = (xxx$ - - to $m - 1$ factors$)\, dx + (xxx$ - - to $m - 1$ factors$)\, dx +$ etc., to m terms; or $dy = x^{m-1}dx + x^{m-1}dx + x^{m-1}dx +$ etc., to m terms. Therefore $dy = mx^{m-1}dx$.

2d. *When the exponent is a positive fraction.* Let $y = x^{\frac{m}{n}}$, $\frac{m}{n}$ being a positive fraction; then $dy = \frac{m}{n}x^{\frac{m}{n}-1}dx$. For involving both members to the nth power we have $y^n = x^m$. Differentiating this as just shown, we have $ny^{n-1}dy = mx^{m-1}dx$.

Now from $y = x^{\frac{m}{n}}$ we have $y^{n-1} = x^{\frac{mn-m}{n}}$. Substituting this in the last it becomes $nx^{\frac{mn-m}{n}} dy = mx^{m-1}dx$; whence $dy = \frac{m}{n}x^{m-1-\frac{mn-m}{n}}dx = \frac{m}{n}x^{\frac{m}{n}-1}dx$. Q. E. D.

3d. *When the exponent is negative.* Let $y = x^{-n}$, n being integral or fractional; then $dy = -nx^{-n-1}dx$. For $y = x^{-n} = \frac{1}{x^n}$, which differentiated by Rule 7, Cor., gives $dy = -\frac{nx^{n-1}dx}{x^{2n}} = -nx^{-n-1}dx$. Q. E. D.

EXAMPLES.

1. Differentiate $y = 3x^2 - 2x + 4$.

SOLUTION.—The result is $dy = 6xdx - 2dx$. Which is thus obtained: By Rule 1, the differential of y is dy. To differentiate the second member we differentiate each term separately according to Rule 4. In differentiating $3x^2$, we observe that the factor 3 is retained in the differential, Rule 2, and the differential of x^2 is, by Rule 8, $2xdx$. Hence, the differential of $3x^2$ is $6xdx$. The differential of $-2x$ is $-2dx$. By Rule 3, the constant 4 disappears from the differential, or its differential is 0.

2. Differentiate $y = 2ax^2 + 4ax^3 - x + m$.

Result, $dy = 4axdx + 12ax^2dx - dx$.

3. Differentiate $y = 5bx^3 - 30x^2 + 4x$.

4. Differentiate $y = Ax^2 + Bx^3 + Cx^4$.

157. SCH.—It is desirable that the pupil not only become expert in writing out the differentials of such expressions as the above, but that he know what the operation signifies. Thus, suppose we have the equation $y = 5x$. This expresses a relation between x and y. Now, if x changes value, y must change also in order to keep the equation true. In this simple case it is easy to see that y must change 5 times as fast as x in order to keep the equation true. This is what differentiation shows. Thus, differentiating, we have $dy = 5dx$. That is, if x takes an infinitesimal increment, y takes an infinitesimal increment equal to 5 times that which x takes; or, in other words, y increases 5 times as fast as x.

Now let us take a case which is not so simple. Let $y=3x^2-2x+4$, and let it be required to find *the relative rate of change of* x *and* y. Differentiating, we have $dy = 6xdx - 2dx = (6x - 2)dx$. This shows that, if x takes an infinitesimal increment represented by dx, y takes one (represented by dy) which is $6x - 2$ times as large; *i. e.*, that y increases $6x - 2$ times as fast as x. Notice that in this case the relative rate of increase of x and y depends on the value of x. Thus, when $x=1$, y is increasing 4 times as fast as x; when $x=2$, y is increasing 10 times as fast as x; when $x=3$, y is increasing 16 times as fast as x; etc.

5. Differentiate $y = x^5 - x^3$, and explain the significance of the result as above. *Result*, $dy = (5x^4 - 3x^2)dx$.

6. In order to keep the relation $2y = 3x^2$ true as x varies, how must y vary in relation to x? What is the relative rate of change when $x = 4$? When $x = 2$? When $x = 1$? When $x = \frac{1}{3}$? When $x = \frac{1}{6}$?

Answers. When $x = 4$, y increases 12 times as fast as x. When $x = \frac{1}{3}$, y increases at the same rate as x. In general y increases $3x$ times as fast as x. When x is less than $\frac{1}{3}$, y increases slower than x.

7 to 12. Differentiate the following: $u = \frac{2x^2}{3y}$; $u = \frac{x^2 - 1}{x^2 + 1}$; $y = x^{\frac{1}{2}}z^2$; $u = x^2y^3 + 6x$; $y = x^5 - 3x^4 + 4x^3 - x^2 + 1$; and $y = \frac{1}{2}x^2 - \frac{2}{3}x^3 + x$.

13 to 17. Differentiate $y = (a^2 + x^3)^5$; $y = (a + x^2)^{\frac{2}{3}}$; $y = (3x - 2)^{\frac{3}{2}}$; $y = (2 - x^2)^{-2}$; and $y = (1 + x)^{-\frac{1}{2}}$.

SUG'S.—Such examples should be solved by considering the entire quantity within the parenthesis as the variable. This is evidently admissible, since any expression which contains a variable is variable when taken as a whole. Thus to differentiate $y = (a + x^2)^{\frac{2}{3}}$, we take the continued product of the exponent ($\frac{2}{3}$), the variable $(a + x^2)$ with its exponent diminished by 1, [*i. e.*, $(a + x^2)^{-\frac{1}{3}}$], and the differential of the variable (*i. e.*, the differential of $a + x^2$, which is $2xdx$). This gives us $dy = \frac{2}{3}(a + x^2)^{-\frac{1}{3}}2xdx$, or $dy = \frac{4}{3}x(a + x^2)^{-\frac{1}{3}}dx = \frac{4xdx}{3\sqrt[3]{a + x^2}}$.

18 to 22. Differentiate $\frac{1}{1 + x}$; $\frac{1}{(1 + x)^2}$; $\frac{1}{(1 + x)^3}$; $-m\frac{1}{(1 + x)^5}$ and $-m\frac{1}{(1 + x)^3}$.

23. In the expression $6x^3$, when x is greater than 1 does the function ($6x^3$) change faster or slower than x? How, when x is less than $\frac{1}{2}$? What does the process of differentiating $6x^3$ signify?

Answer to the last. Finding the relative rate of change of $6x^3$ and x, or finding what increment $6x^3$ takes when x takes the increment dx.

Or, in still other words, finding the difference between two consecutive states of $6x^3$, and hence the relation between an infinitesimal increment of x and the corresponding increment of $6x^3$.

SECTION II.

INDETERMINATE COEFFICIENTS.

158. *Indeterminate Coefficients* are coefficients assumed in the demonstration of a theorem or the solution of a problem, whose values are not known at the outset, but are to be determined by subsequent processes.

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159. *Prop.*—*If* $A + Bx + Cx^2 + Dx^3 + etc. = A' + B'x + C'x^2 + D'x^3 + etc.$, *in which* x *is a variable* and the coefficients* A, B, A′, B′, *etc. are constants, the coefficients of the like powers of* x *are equal to each other. That is,* $A = A'$ (*these being the coefficients of* x^0), $B = B'$, $C = C'$, *etc.*

DEM.—Since the equation is true for *any* value of x, it is true for $x=0$. Substituting this value, we have $A=A'$. Now as A and A' are constant, they have the same values whatever the value assigned to x. Hence for *any* value of x, $A=A'$. Again, dropping A and A', we have $Bx + Cx^2 + Dx^3 +$ etc. $= B'x + C'x^2 + D'x^3 +$ etc., which is true for any value of x. Dividing by x, we obtain $B + Cx + Dx^2 +$ etc. $= B' + C'x + D'x^2 +$ etc., likewise true for any value of x. Making $x = 0$, $B = B'$, as before. In this manner we may proceed, and show that $C = C'$, $D = D'$, etc. Q. E. D.

160. COR.—*If* $A + Bx + Cx^2 + Dx^3 + etc. = 0$, *is true for all values of* x, *each of the coefficients* A, B, C, *etc., is* 0.

For we may write $A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ etc. $= 0 + 0x + 0x^2 + 0x^3 + 0x^4 + 0x^5 +$ etc. Whence by the proposition $A = 0, B = 0, C = 0$, etc.

DEVELOPMENT OF FUNCTIONS BY MEANS OF INDETERMINATE COEFFICIENTS.

161. A Function is said to be ***Developed*** when the indicated operations are performed; or, more properly, when it is transformed into an equivalent series of terms following some general law.

ILL'S.—Division affords a method of developing some forms of functions.

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* Saying that x is a variable, is equivalent to saying that the equation must be true for *any* value of x. This is an essential thing in this discussion. The members of such an equation are sometimes said to be *Identically* equal.

Thus $y=\frac{1}{1-x}$ when developed by division becomes $y=1+x+x^2+x^3+$ etc. The binomial formula (COMPLETE SCHOOL ALGEBRA, ***195***, or ***168*** of this treatise) is a formula for developing a binomial. Thus $y=(a+x)^5$ when developed becomes $y=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$. The subject is one of great importance in mathematics, and the method of Indeterminate Coefficients forms the basis of most that is valuable upon it.

EXAMPLES.

1. Develop $\frac{1-x}{1+x+x^2}$ into a series by the method of Indeterminate Coefficients.

SOLUTION.—Assume $\frac{1-x}{1+x+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+$ etc. Clearing of fractions,

$$\begin{array}{r|r|r|r|l} 1-x=A+B & x+C & x^2+D & x^3+E & x^4+\text{ etc.} \\ +A & +B & +C & +D & +\text{ etc.} \\ & +A & +B & +C & +\text{ etc.} \end{array}$$

Equating the coefficients of the corresponding powers of x by (***159***), we have the following equations from which to find the values of A, B, C, D, etc.: $A=1$; $A+B=-1$; $A+B+C=0$; $B+C+D=0$; $C+D+E=0$. Solving these, we have $A=1$, $B=-2$, $C=1$, $D=1$, and $E=-2$.

Substituting these in the assumed development, we have

$$\frac{1-x}{1+x+x^2}=1-2x+x^2+x^3-2x^4+\text{ etc.}$$

This can readily be verified by actual division.

2. Develop, or expand into a series $(a^2-x^2)^{\frac{3}{2}}$ by means of Indeterminate Coefficients.

SOLUTION.—Assume

$$(a^2-x^2)^{\frac{3}{2}}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+\text{ etc.}$$

Squaring both members and expanding $(a^2-x^2)^3$, we have

$$\begin{array}{r|r|r|r|r|r|l} a^6-3a^4x^2+3a^2x^4-x^6=A^2+AB & x+AC & x^2+AD & x^3+AE & x^4+AF & x^5+AG & x^6+\text{etc.} \\ +AB & +B^2 & +BC & +BD & +BE & +BF & +\text{etc.} \\ & +AC & +BC & +C^2 & +CD & +CE & +\text{etc.} \\ & & +AD & +BD & +CD & +D^2 & +\text{etc.} \\ & & & +AE & +BE & +CE & +\text{etc.} \\ & & & & +AF & +BF & +\text{etc.} \\ & & & & & +AG & +\text{etc.} \end{array}$$

Equating the coefficients of the corresponding powers of x, we find $A^2=a^6$, or $A=a^3$; $2AB=0$, whence $B=0$; $2AC+B^2=-3a^4$, whence $C=-\frac{3}{2}a$; $2(AD+BC)=0$, whence $D=0$; $2(AE+BD)+C^2=3a^2$, whence $E=\frac{3}{8a}$;

$2(AF+BE+CD)=0$, whence $F=0$; and in like manner $G=\frac{1}{16a^3}$, etc. (If the expansion of the second member had been carried farther, each of the succeeding coefficients would be equated with 0, as there are no terms in the first member containing higher powers of x than the 6th.) Substituting the values of A, B, C, D, etc, as now found, we have $(a^2-x^2)^{\frac{3}{2}}=a^3-\frac{3}{2}ax^2+\frac{3x^4}{8a}+\frac{x^6}{16a^3}+$ etc.

3. Expand, or develop $(1-x^2)^{\frac{1}{2}}$ by means of Indeterminate Coefficients. Also $\frac{x}{x+1}$, $\frac{d}{b-ax}$, and $\frac{1}{(1-x)^{\frac{1}{2}}}$.

SUG.—To expand the last, put the expression equal to the usual series, square both members, and then clear of fractions.

162. SCH.—In using the method of Indeterminate Coefficients, as the series $A+Bx+Cx^2+$ etc., is merely hypothetical at the outset, we must carefully observe whether the subsequent processes develop any inconsistency. For example, perhaps a particular expression will not develop in the form assumed. If so, some inconsistency will appear in the process. Thus, were we to attempt to develop $\frac{2}{x^2-x^3}$ by assuming $\frac{2}{x^2-x^3}=A+Bx+Cx^2+Dx^3+$ etc., we should find, after clearing of fractions, that the first member had only the term 2, which is $2x^0$; and as there would be no corresponding term in the second member, we should have to write $2=0$, which is absurd. In general, we observe that, when we equate the coefficients, the second, or assumed member, must have a term containing as low a power of the variable as the lowest in the first member. This may be secured either by putting the expression to be developed into a proper form before assuming the series, or by assuming a series of proper form. Thus, in the above case, we may write for $\frac{2}{x^2-x^3}$, $\frac{1}{x^2}\cdot\frac{2}{1-x}$, and then develop $\frac{2}{1-x}$ by assuming $\frac{2}{1-x}=A+Bx+Cx^2+Dx^3+$ etc., and finally multiplying by $\frac{1}{x^2}$; or it may be developed by assuming $\frac{2}{x^2-x^3}=Ax^{-2}+Bx^{-1}+Cx^0+Dx+Ex^2+$ etc.

4. Expand $\frac{x^4-3x^2+2}{x^2-x+1}$ by the method of Indeterminate Coefficients. Also $\frac{1+2x}{1-3x^2}$. Also $\frac{1-x}{2x^2+3x^3}$.

5. Expand $\sqrt[3]{1-x}$. Also $(1+x)^{\frac{2}{3}}$.

Decomposition of Fractions by means of Indeterminate Coefficients.

163. For certain purposes, especially in the Integral Calculus, it is often necessary to decompose a fraction into partial fractions. There are three principal cases.

164. Case 1.—*A fraction which is a function of a single variable, whose numerator is of lower dimensions than its denominator, and whose denominator is resolvable into* n REAL *and* UNEQUAL *factors of the first degree, can be decomposed into* n *partial fractions of the form* $\frac{\mathrm{A}}{\mathrm{x+a}} + \frac{\mathrm{B}}{\mathrm{x+b}} + \frac{\mathrm{C}}{\mathrm{x+c}} - - - - - - \frac{\mathrm{N}}{\mathrm{x+n}}$, x + a, x + b, x + c, - - - - - x + n *being the factors of the denominator.*

Dem.—Assume $\frac{f(x)}{\varphi(x)}$* $= \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c} - - - - - - \frac{N}{x+n}$, in which $f(x)$ is of lower dimensions† than $\varphi(x)$, and $\varphi(x) = (x+a)(x+b)(x+c)$ - - - - - $(x+n)$.‡ Reducing the partial fractions $\frac{A}{x+a}$, $\frac{B}{x+b}$, etc., to forms having a common denominator, this denominator will be the product of all the denominators $x+a$, $x+b$, $x+c$, etc., and hence will be $\varphi(x)$, and each numerator will contain one less of these factors than the common denominator, and hence will be of the $(n-1)$th degree, the denominator being of the nth degree.§ Then, as the denominators of both members will be equal, the numerators will also be equal. Placing them so, we can find the values of the indeterminate coefficients A, B, C, etc., by the principle in (***159***). The necessity for having $f(x)$ of lower dimensions than $\varphi(x)$ is the same as is pointed out in (***162***). Thus, if $f(x)$ contained a term like $5x^5$ while $\varphi(x)$ contained none higher than $2x^2$, we should be required to write $5 = 0$, as there would be no term in the second member having an x^5 in it. Finally, having obtained the values of A, B, C, etc., we can substitute them in $\frac{A}{x+a}$, $\frac{B}{x+b}$, $\frac{C}{x+c}$, etc., and have the partial fractions sought.

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165. Case 2.—*A fraction which is a function of a single variable, whose numerator is of lower dimensions than its denominator, and whose denominator is resolvable into* n REAL *and* EQUAL *factors*

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* See (***139, 140***).

† That is, does not contain so high a power of x.

‡ The proposition assumes that $\varphi(x)$ is resolvable into n real and unequal factors of the first degree.

§ That is, containing x to the nth power, and no higher power.

of the first degree, can be decomposed into n *partial fractions of the form* $\frac{A}{(x+a)^n} + \frac{B}{(x+a)^{n-1}} + \frac{C}{(x+a)^{n-2}} - - - - - - \frac{N}{x+a}$, $x + a$ *being one of the equal factors of the denominator.*

DEM.—Assume $\frac{f(x)}{\varphi(x)} = \frac{A}{(x+a)^n} + \frac{B}{(x+a)^{n-1}} + \frac{C}{(x+a)^{n-2}} - - - - \frac{N}{x+a}$, in which $f(x)$ is of lower dimensions than $\varphi(x)$, and $\varphi(x) = (x+a)^n$. Reducing the partial fractions to forms having the common denominator $(x+a)^n$ (*i. e.* $\varphi(x)$), and placing the numerators of the members equal, we find that the second member is not of lower dimensions with respect to the variable x, than the first member, since the numerator of the fraction $\frac{N}{x+a}$ will contain the highest power of x of any of the terms, and this will have no higher power than x^{n-1}, as $(x+a)^{n-1}$ is the factor by which the terms of the fraction $\frac{N}{x+a}$ will be multiplied in the reduction. Hence, we can find the values of A, B, C, etc., by (***159***), and these substituted in the assumed series will give the required partial fractions.

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166. CASE 3.—*A fraction which is a function of a single variable, whose numerator is of lower dimensions than its denominator, and whose denominator is resolvable into* n REAL *and equal* QUADRATIC *factors, can be decomposed into* n *partial fractions of the form* $\frac{Ax+B}{[(x+a)^2+b^2]^n} + \frac{Cx+D}{[(x+a)^2+b^2]^{n-1}} + \frac{Ex+F}{[(x+a)^2+b^2]^{n-2}} - - - - - - \frac{Mx+N}{(x+a)^2+b^2}$, $(x+a)^2+b^2$ *being one of the equal factors of the denominator.*

DEM.—Assume

$$\frac{f(x)}{\varphi(x)} = \frac{Ax+B}{[(x+a)^2+b^2]^n} + \frac{Cx+D}{[(x+a)^2+b^2]^{n-1}} + \frac{Ex+F}{[(x+a)^2+b^2]^{n-2}} - - - - \frac{Mx+N}{(x+a)^2+b^2}.$$

Bringing the terms of the second member to a common denominator $\varphi(x)$, or $[(x+a)^2+b^2]^n$, we find that the highest power of x involved in the numerators is x^{2n-1}, which will arise in multiplying $Mx+N$ by $[(x+a)^2+b^2]^{n-1}$. But, as $f(x)$ is of lower dimensions than $\varphi(x)$, and $\varphi(x)$ is of $2n$ dimensions, the numerator of the second member will not be of lower dimensions than $f(x)$, and hence equating them, the values of A, B, C, etc., can be determined and substituted in the assumed series of partial fractions.

167. SCH.—When the denominator of the fraction to be decomposed is composed of factors of two or more of the forms referred to in the three given cases, the *forms* of the assumed partial fractions must be made to correspond. Thus were it required to decompose $\frac{x^3-3x^2+5}{x(x-b)(x+a)^3(x^2+a^2)^2}$, the assumed partial fractions would be $\frac{A}{x}+\frac{B}{x-b}+\frac{C}{(x+a)^3}+\frac{D}{(x+a)^2}+\frac{E}{x+a}+\frac{Fx+G}{(x^2+a^2)^2}$ $+\frac{Hx+I}{x^2+a^2}$.

EXAMPLES.

1. Decompose $\frac{x^2-2}{x-x^3}$ into partial fractions.

SOLUTION.—Assume $\frac{x^2-2}{x-x^3}=\frac{A}{x}+\frac{B}{1-x}+\frac{C}{1+x}$, x, $1-x$, and $1+x$ being the unequal factors of $x-x^3$ (**115, 117**). Bringing the terms of the second member to a common denominator, we have

$$\frac{x^2-2}{x-x^3}=\frac{A-Ax^2+Bx+Bx^2+Cx-Cx^2}{x(1-x)(1+x)}.$$

Hence $x^2-2=A+(B+C)x+(B-A-C)x^2$; from which we get $A=-2$, $B+C=0$, and $B-A-C=1$. Solving these equations we find $A=-2$, $B=-\frac{1}{2}$, and $C=\frac{1}{2}$. These values inserted in the assumed forms give

$$\frac{x^2-2}{x-x^3}=\frac{-2}{x}-\frac{\frac{1}{2}}{1-x}+\frac{\frac{1}{2}}{1+x}=-\frac{2}{x}-\frac{1}{2(1-x)}+\frac{1}{2(1+x)}.$$

2 to 6. Decompose the following: $\frac{x+3}{x^2-x-2}$; $\frac{x+1}{x(x-2)}$; $\frac{x+1}{x^2-7x+12}$; $\frac{3x-5}{x^2-6x+8}$; and $\frac{x^2}{x^3+6x^2+11x+6}$.

SUG.—In case the factors of the denominator are not readily discerned, place the denominator equal to 0 and resolve the equation. Thus the last example gives $x^3+6x^2+11x+6=0$. From which we have $x=-1$, -2, and -3 (**119**), and the factors are $x+1$, $x+2$, and $x+3$.

7 to 11. Decompose the fractions $\frac{3x^2-7x+6}{(x-1)^3}$; $\frac{2+3x+x^2}{x^3+9x^2+27x+27}$; $\frac{1}{x^3(1-x^2)(1+x)}$; $\frac{1}{x^4-1}$; and $\frac{1}{(x-2)^2(x+3)^2}$.

12 to 18. Decompose $\frac{x^2-2x+3}{(x^2+1)^2}$; $\frac{3x^5-x^4-10x^3+15x^2+2x-8}{x(x^2-2)^2(x-1)}$; $\frac{x^2-x+1}{x^2(x+1)}$; $\frac{1}{x^6-1}$; $\frac{1}{a^4-x^4}$; $\frac{1}{x^2-(a+b)x+ab}$; and $\frac{6x^2-4x-6}{x^3-6x^2+11x-6}$.

SECTION III.

THE BINOMIAL FORMULA.

168. Theorem.—*Letting* x *and* y *represent any quantities whatever* (i. e. *be variables*) *and* m *any constant,*

$$(x+y)^m = x^m + mx^{m-1}y + \frac{m(m-1)}{\lfloor 2}x^{m-2}y^2 + \frac{m(m-1)(m-2)}{\lfloor 3^*}x^{m-3}y^3$$
$$+ \frac{m(m-1)(m-2)(m-3)}{\lfloor 4^*}x^{m-4}y^4 + \text{etc.}$$

DEM.—We may write $(x+y)^m = x^m\left(1+\frac{y}{x}\right)^m$. Now put $\frac{y}{x} = z$ and assume

$$(1+z)^m = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}, \quad (1)$$

in which A, B, C, etc., are indeterminate coefficients independent of z (*i. e.* constants), and are to be determined. To determine these coefficients we proceed as follows :

Differentiating (1), we have

$$m(1+z)^{m-1}dz = Bdz + 2Czdz + 3Dz^2dz + 4Ez^3dz + 5Fz^4dz + \text{etc.}$$

Dividing by dz, we have

$$m(1+z)^{m-1} = B + 2Cz + 3Dz^2 + 4Ez^3 + 5Fz^4 + \text{etc.} \quad (2)$$

Differentiating (2) and dividing by dz, we have

$$m(m-1)(1+z)^{m-2} = 2C + 2\cdot 3Dz + 3\cdot 4Ez^2 + 4\cdot 5Fz^3 + \text{etc.} \quad (3)$$

Differentiating (3) and dividing by dz, we have

$$m(m-1)(m-2)(1+z)^{m-3} = 2\cdot 3D + 2\cdot 3\cdot 4Ez + 3\cdot 4\cdot 5Fz^2 + \text{etc.} \quad (4)$$

Differentiating (4) and dividing by dz, we have

$$m(m-1)(m-2)(m-3)(1+z)^{m-4} = 2\cdot 3\cdot 4E + 2\cdot 3\cdot 4\cdot 5Fz + \text{etc.} \quad (5)$$

Differentiating (5) and dividing by dz, we have

$$m(m-1)(m-2)(m-3)(m-4)(1+z)^{m-5} = 2\cdot 3\cdot 4\cdot 5F + \text{etc.} \quad (6)$$

We have now gone far enough to enable us to determine the coefficients A, B, C, D, E, and F, and doubtless to determine the law of the series.

As all the above equations are to be true for all values of z, and as the coeffi-

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* This form is read "factorial 3," "factorial 4," etc. ; and signifies the product of the natural numbers from 1 to 3, 1 to 4, etc.

cients A, B, C, etc., are constants, *i. e.*, have the same values for one value of z as for another, if we can determine their values for one value of z, these will be their values in all cases. Now, making $z=0$, we have from (1) $A=1$; from (2), $B=m$; from (3), $C=\frac{m(m-1)}{\lfloor 2}$ (the factor 1 being introduced into the denominator for the sake of symmetry); from (4), $D=\frac{m(m-1)(m-2)}{\lfloor 3}$; from (5), $E=\frac{m(m-1)(m-2)(m-3)}{\lfloor 4}$; from (6), $F=\frac{m(m-1)(m-2)(m-3)(m-4)}{\lfloor 5}$.

These values substituted in (1) give

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{\lfloor 2}z^2 + \frac{m(m-1)(m-2)}{\lfloor 3}z^3 + \frac{m(m-1)(m-2)(m-3)}{\lfloor 4}z^4 + \frac{m(m-1)(m-2)(m-3)(m-4)}{\lfloor 5}z^5 + \text{etc.}$$

Finally, replacing z by its value $\frac{y}{x}$, we have

$$(x+y)^m = x^m\left(1+\frac{y}{x}\right)^m = x^m\left\{1 + m\frac{y}{x} + \frac{m(m-1)}{\lfloor 2}\frac{y^2}{x^2} + \frac{m(m-1)(m-2)}{\lfloor 3}\frac{y^3}{x^3} + \frac{m(m-1)(m-2)(m-3)}{\lfloor 4}\frac{y^4}{x^4} + \frac{m(m-1)(m-2)(m-3)(m-4)}{\lfloor 5}\frac{y^5}{x^5} + \text{etc.}\right\}$$

$$= x^m + mx^{m-1}y + \frac{m(m-1)}{\lfloor 2}x^{m-2}y^2 + \frac{m(m-1)(m-2)}{\lfloor 3}x^{m-3}y^3 + \frac{m(m-1)(m-2)(m-3)}{\lfloor 4}x^{m-4}y^4 + \frac{m(m-1)(m-2)(m-3)(m-4)}{\lfloor 5}x^{m-5}y^5 + \text{etc.}$$

169. Cor. 1.—*The* n*th, or general term of the series is*

$$\frac{m(m-1)(m-2)\text{ - - - - - }(m-n+2)}{\lfloor n-1}x^{m-n+1}y^{n-1}.$$

For we observe that the last factor in the numerator of the coefficient of any particular term is m — the number of the term less 2, *i. e.*, for the nth term, $m-(n-2)$, or $m-n+2$; and the last factor in the denominator is the number of the term -1, *i. e.*, for the nth term, $n-1$. The exponent of x in any particular term is m — the number of the term less 1, *i. e.*, for the nth term, $m-(n-1)$, or $m-n+1$; and the exponent of y in any term is one less than the number of the term, *i. e.*, for the nth term, $n-1$.

170. Def.—In a series the ***Scale of Relation*** is the relation which exists between any term or set of terms and the next term or set of terms.

171. Cor. 2.—*The scale of relation in the binomial series is* $\left(\frac{m+1}{n}-1\right)\frac{y}{x}$, *since the* n*th term multiplied by this produces the* $(n+1)$*th term.*

This is readily seen by inspecting the series, or by writing the $(n+1)$th term and dividing it by the nth. Thus, substituting in the general term as given above, $n+1$ for n, we have $\frac{m(m-1)(m-2) \text{ - - - - - } (m-n+1)}{\lfloor n}x^{m-n}y^n$, as the $(n+1)$th term. This divided by the nth, or preceding term,* gives $\frac{m-n+1}{n}\,\frac{y}{x}$, or $\left(\frac{m+1}{n}-1\right)\frac{y}{x}$.

EXAMPLES.

1 to 6. Expand the following: $(a-b)^6$; $(x-y)^7$; $(a-x)^m$; $(1+x)^4$; $(1-y)^5$; $(1-y)^n$.

7 to 11. Expand $(x+y)^{-2}$; $(x-y)^{-3}$; $(a-x)^{-1}$; $\frac{1}{(a+x)^4}$; $\frac{1}{x+y}$, or $(x+y)^{-1}$.

[NOTE.—For practical suggestions in the use of this theorem, see COMPLETE SCHOOL ALGEBRA, pages 148–154, or PART I. of this volume, pages 58, 59.]

12. Expand $(a+x)^5$ by using the scale of relation.

SOLUTION.—The scale of relation $\left(\frac{m+1}{n}-1\right)\frac{y}{x}$ becomes in this case $\left(\frac{5+1}{n}-1\right)\frac{x}{a}$. Now the first term is a^5. To obtain the next $n=1$, whence the scale of relation $5\frac{x}{a}$. Multiplying a^5 by this scale of relation, we find the second term $5a^4x$. For the next the scale of relation is $2\frac{x}{a}$. Hence the 3d term is $10a^3x^2$. For the next the scale of relation is $\frac{x}{a}$, giving for the 4th term $10a^2x^3$. For the 5th term the scale of relation is $\left(\frac{6}{4}-1\right)\frac{x}{a}$ or $\frac{1}{2}\frac{x}{a}$, giving for this term $5ax^4$. For the 6th term the scale of relation equals $\left(\frac{6}{5}-1\right)\frac{x}{a}$ or $\frac{1}{5}\frac{x}{a}$, giving x^5. For the 7th term the scale of relation is $\left(\frac{6}{6}-1\right)\frac{x}{a}$ or 0. Hence the series terminates.

13. Expand $(m-n)^{-5}$ by using the scale of relation, and also by the general formula.

14 to 17. Expand $(1-a^2)^{\frac{1}{2}}$; $(2+x^3)^{\frac{2}{3}}$; $(x-y)^{-\frac{1}{2}}$; $(a+x)^{-\frac{3}{4}}$.

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* The numerator of the coefficient of the preceding, or nth term, contains all the factors of the numerator of the $(n+1)$th except $m-n+1$, as the factor in the $(n+1)$th preceding $m-n+1$ is $m-n+2$, etc. Similarly in the denominator.

18 to 20. Expand $(a^2 - x^2)^{\frac{1}{3}}$; $(3a - x^2)^{-3}$; $(a^2 + c^{\frac{1}{2}})^4$.

SUG'S.—In cases in which the terms of the binomial are not single letters or figures, it will be best to substitute single letters, expand and then replace the values. Thus, to expand $(x^3-3a)^{-\frac{1}{2}}$, put $x^3 = y$, and $3a = b$, and expand $(y-b)^{-\frac{1}{2}}$; and in this expansion restore the values of y and b. In like manner the formula may be applied to any polynomial. Thus, to expand $(1 - x^2 + 3y)^3$, put $(1 - x^2) = z$, and $3y = u$, expand $(z + u)^3$, and then restore the values.

21. Expand $\dfrac{a}{\sqrt{b^2 - c^2x^2}}$ into a series.

SUG'S. $\dfrac{a}{\sqrt{b^2-c^2x^2}} = a(b^2-c^2x^2)^{-\frac{1}{2}}$. Put $b^2 = v$, and $c^2x^2 = y$, and expand $(v-y)^{-\frac{1}{2}}$, etc. The result is

$$\frac{a}{\sqrt{b^2 - c^2x^2}} = \frac{a}{b}\left\{1 + \frac{1}{2}\frac{c^2x^2}{b^2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{c^4x^4}{b^4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{c^6x^6}{b^6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{c^8x^8}{b^8} + \text{etc.}\right\}$$

22. What is the 4th term of the development of $(a^2 + z)^{\frac{1}{2}}$? (See ***169***.)

SUG.—The general term is $\dfrac{m(m-1) \cdot \cdot \cdot \cdot (m-n+2)}{\lfloor n-1}x^{m-n+1}y^{n-1}$. In this case $m = \frac{1}{2}$, $n = 4$, $x = a^2$, $y = z$. Whence the 4th term is $\dfrac{z^3}{16a^5}$.

23. What is the 7th term of $(a^2 - b^2)^{\frac{2}{3}}$? The 10th term?

SECTION IV.

LOGARITHMS.

172. A Logarithm is the exponent by which a fixed number is to be affected in order to produce any required number. The fixed number is called the *Base* of the System.

ILL.—Let the *Base* be 3: then the logarithm of 9 is 2; of 27, 3; of 81, 4; of 19683, 9; for $3^2 = 9$; $3^3 = 27$; $3^4 = 81$; and $3^9 = 19683$. Again, if 64 is the base, the logarithm of 8 is $\frac{1}{2}$, or .5, since $64^{\frac{1}{2}}$, or $64^{.5} = 8$; *i.e.*, $\frac{1}{2}$, or .5 is the exponent by which 64, the base, is to be affected in order to produce the number 8. So, also, 64 being the base, $\frac{1}{3}$, or .333+ is the logarithm of 4, since $64^{\frac{1}{3}}$, or $64^{.333+} = 4$; *i. e.*, $\frac{1}{3}$, or .333+ is the exponent by which 64, the base, is to be affected in order to produce the number 4. Once more, since $64^{\frac{2}{3}}$, or $64^{.666+} = 16$, $\frac{2}{3}$, or .666+ is the logarithm of 16, if the base is 64. Finally, $64^{-\frac{1}{2}}$, or $64^{-.5} = \frac{1}{8}$, or .125; hence $-\frac{1}{2}$, or $-.5$ is the logarithm of $\frac{1}{8}$, or .125, when the base is 64. In like manner, with the same base, $-\frac{1}{3}$, or $-.333+$ is the logarithm of $\frac{1}{4}$, or .25.

173. COR.—*Since any number with* 0 *for its exponent is* 1, *the logarithm of* 1 *is* 0, *in all systems. Thus* $10^0 = 1$, *whence* 0 *is the logarithm of* 1, *in a system in which the base is* 10.

174. A System of Logarithms is a scheme by which all numbers can be represented, either exactly or approximately, by exponents by which a fixed number (the base) can be affected.

175. There are *Two Systems* of Logarithms in common use, called, respectively, the *Briggean* or *Common* System, and the *Napierian* or *Hyperbolic* System.* The base of the former is 10, and of the latter 2.71828+. In the present treatise we shall confine our attention to systems whose bases are greater than 1.

176. COR. 1.—*Neither* 1 *nor any negative number can be used as the base of a system of logarithms.*

For *all* numbers cannot be represented either exactly or approximately by exponents of such numbers. Thus with 1 as a base we can represent no other number than 1 by its exponents, for 1 with *any* exponent is 1. Moreover, with a negative base the logarithms which were odd numbers would represent negative numbers, and those which were even numbers would represent positive numbers. For example, with -2 as a base, 3 might be considered as the logarithm of -8, since $(-2)^3 = -8$; but no number could be found as a logarithm to correspond to 8 (*i. e.* +8), since -2 cannot be affected with any exponent which will produce 8.

177. One of the most important uses of logarithms is to facilitate the multiplication, division, involution, and the extraction of roots of large numbers. These processes are performed upon the following principles:

178. Prop. 1.—*The logarithm of the product of two numbers is the sum of their logarithms.*

DEM.—Let a be the base of the system. Let m and n be any two numbers whose logarithms are x and y respectively. Then by definition $a^x = m$, and $a^y = n$.

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* The common system is the one used for practical purposes, and the only one of which there are tables in common use. Napierian logarithms are usually implied in abstract mathematical discussion.

Multiplying the corresponding members of these equations together we have $a^{x+y}=mn$. Whence $x+y$ is the logarith of mn. Q. E. D.

179. Prop. 2.—*The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

DEM.—Let a be the base of the system, and m and n any two numbers whose logarithms are, respectively, x, and y. Then by definition we have $a^x=m$, and $a^y=n$. Dividing, we have $a^{x-y}=\frac{m}{n}$. Whence $x-y$ is the logarithm of $\frac{m}{n}$. Q. E. D.

180. Prop. 3.—*The logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

DEM.—Let a be the base, and x the logarithm of m. Then $a^x=m$; and raising both to any power, as the zth, we have $a^{xz}=m^z$. Whence xz is the logarithm of the zth power of m. Q. E. D

181. Prop. 4.—*The logarithm of any root of a number is the logarithm of the number divided by the number expressing the degree of the root.*

DEM.—Let a be the base, and x the logarithm of m. Then $a^x=m$. Extracting the zth root we have $a^{\frac{x}{z}}=\sqrt[z]{m}$. Whence $\frac{x}{z}$ is the logarithm of $\sqrt[z]{m}$. Q. E. D.

182. It is evident that in any system, the logarithms of most numbers will not be expressed in integers. Thus in the common system the logarithm of 100 is 2, and of 1000 3; hence the logarithm of any number between 100 and 1000 is between 2 and 3, *i. e.* 2 and some fraction. This fraction is usually written as a decimal fraction, and, as we shall see more clearly hereafter, can in general be expressed only approximately.

183. The Integral Part of a logarithm is called the ***Characteristic,*** and the decimal part the ***Mantissa.***

184. Prop.—*The Mantissa of the logarithm of a decimal fraction, or of a mixed number, is the same as the mantissa of the number considered as integral.**

* Usually, in speaking of logarithms, if no particular system is mentioned, the common system is to be understood as meant, especially when practical numerical operations are referred to.

DEM.—It will be found hereafter that log 2845672=6.454185. Now this means that $10^{6.454185}=2845672$. Dividing by 10 successively we have

$$10^{5.454185} = 284567.2, \quad \text{or } \log 284567.2 = 5.454185,$$
$$10^{4.454185} = 28456.72, \quad \text{or } \log 28456.72 = 4.454185,$$
$$10^{3.454185} = 2845.672, \quad \text{or } \log 2845.672 = 3.454185,$$
$$10^{2.454185} = 284.5672, \quad \text{or } \log 284.5672 = 2.454185,$$
$$10^{1.454185} = 28.45672, \quad \text{or } \log 28.45672 = 1.454185,$$
$$10^{0.454185} = 2.845672, \quad \text{or } \log 2.845672 = 0.454185.$$

Now if we continue the operation of division, only writing $0.454185 - 1$, $\bar{1}.454185$, meaning by this that the characteristic is negative and the mantissa positive, and the subtraction not performed, we have

$$10^{\bar{1}.454185} = .2845672, \quad \text{or } \log .2845672 = \bar{1}.454185,$$
$$10^{\bar{2}.454185} = .02845672, \quad \text{or } \log .02845672 = \bar{2}.454185,$$
$$10^{\bar{3}.454185} = .002845672, \quad \text{or } \log .002845672 = \bar{3}.454185,$$

etc., etc. Q. E. D.

185. COR. 1.—*The characteristic of the logarithm of an integral number, or of a mixed integral and decimal fractional number, is one less than the number of integral places in the number.*

The characteristic of the logarithm of a number entirely decimal fractional is negative and numerically one greater than the number of 0's immediately following the decimal point.

Thus the characteristic of the logarithm of any number between 1 and 10 is 0, between 10 and 100 1, between 100 and 1000 2, etc. Or let it be asked, "What is the characteristic of the logarithm of 5126?" Now this number lies between 1000 and 10000, hence its logarithm lies between 3 and 4, and is, therefore, 3 and some fraction.

Again, as to the numerical value of the characteristic of the logarithm of a number wholly decimal fractional, consider that $10^{-1} = \frac{1}{10} = .1$; $10^{-2} = \frac{1}{100} = .01$; $10^{-3} = \frac{1}{1000} = .001$. Thus it appears that any number between 1 and .1, *i. e.*, any number expressed by a decimal fraction having a significant figure in tenth's place, as .2564, .846, .1205, etc., will have its logarithm between 0 (the logarithm of 1) and -1 (the logarithm of .1). Hence such a logarithm will be $-1 +$ *some fraction* (the mantissa). In like manner, any number between .1 and .01, *i. e.*, any decimal fraction whose first significant figure is in 100th's place, as .02568, .0956, .01203, etc., will have for its logarithm $-2 +$ some fraction.

186. COR. 2.—*The common logarithm of* 0 *is* $-\infty$.

Since a number less than unity has a negative characteristic, and this characteristic increases *numerically* as the number decreases, when the number decreases to 0, the logarithm increases *numerically* to ∞. Hence $\log 0 = -\infty$. To illustrate, $\log .1 = \bar{1}$, $\log .01 = \bar{2}$, $\log .001 = \bar{3}$, $\log .0001 = \bar{4}$. Hence when the number of 0's becomes infinite, and the number therefore 0, we have $\log 0 = -\infty$.

Computation of Logarithms.

187. The Modulus of a system of logarithms is a constant factor which depends upon the base of the system and characterizes the system.

188. Prop.—*The differential of the logarithm of a number is the differential of the number multiplied by the modulus of the system, divided by the number;*

Or, in the Napierian system, the modulus being 1, *the differential of the logarithm of a number is the differential of the number divided by the number.*

Dem.—Let x represent *any* number, *i. e.* be a variable, and n be a constant such that $y=x^n$. Then $\log y=n \log x$ (*180*). Differentiating $y=x^n$, we have $dy=nx^{n-1}dx$; whence

$$n=\frac{dy}{x^{n-1}dx}=\frac{dy}{\frac{x^n}{x}dx}=\frac{dy}{\frac{y}{x}dx}=\frac{\frac{dy}{y}}{\frac{dx}{x}}. \qquad (1)$$

Again, whatever the differentials of $\log y$ and $\log x$ are, n being a constant factor, we shall have the differential of $\log y$ equal to n times the differential of $\log x$, which may be written

$$d(\log y)=n\cdot d(\log x), \text{ whence } n=\frac{d(\log y)}{d(\log x)}. \qquad (2)$$

Now equating the values of n as represented in (1) and (2), we have $\frac{d(\log y)}{d(\log x)}=\frac{\frac{dy}{y}}{\frac{dx}{x}}$. Whence $d(\log y)$ bears the same ratio to $\frac{dy}{y}$, as $d(\log x)$ does to $\frac{dx}{x}$. Let m be this ratio. Then $d(\log y)=\frac{mdy}{y}$, and $d(\log x)=\frac{mdx}{x}$.

We are now to show that m is constant and depends on the base of the system.

To do this, take $y=z^{n'}$, from which we can find as above $n'=\frac{d(\log y)}{d(\log z)}=\frac{\frac{dy}{y}}{\frac{dz}{z}}$. Now as m is the ratio of $d(\log y)$ to $\frac{dy}{y}$, it is also the ratio of $d(\log z)$ to $\frac{dz}{z}$; and $d(\log z)=\frac{mdz}{z}$. Thus we see that in any case the same ratio exists between the differential of the logarithm of a number and the differential of the number divided by the number. Therefore m is a constant factor.

That m depends upon the base of the system is evident, since in a system of logarithms the only quantities involved are the number, its logarithm, and the base. Of these the two former are variables; whence, as the base is the only constant involved in the scheme, m is a function of the base.*

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189. Prob.—To produce the logarithmic series.

SOLUTION.—The logarithmic series, which is the foundation of the usual method of computing logarithms, and of much of the theory of logarithms, is the development of log $(1 + x)$. To develop log $(1 + x)$, assume

$$\log (1 + x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.,} \qquad (1)$$

in which x is a variable, and A, B, C, etc., are constants.

Differentiating (1), we have

$$\frac{mdx\dagger}{1 + x} = Bdx + 2Cxdx + 3Dx^2dx + 4Ex^3dx + 5Fx^4dx + \text{etc.}$$

Dividing by dx,

$$\frac{m}{1 + x} = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \text{etc.} \qquad (2)$$

Differentiating (2), and dividing by dx, we have

$$-m\frac{1}{(1 + x)^2} = 2C + 2\cdot 3Dx + 3\cdot 4Ex^2 + 4\cdot 5Fx^3 + \text{etc.} \qquad (3)$$

Differentiating (3), and dividing by 2 and by dx, we have

$$m\frac{1}{(1 + x)^3} = 3D + 3\cdot 4Ex + 2\cdot 3\cdot 5Fx^2 + \text{etc.} \qquad (4)$$

Differentiating (4), and dividing by 3 and dx, we have

$$-m\frac{1}{(1 + x)^4} = 4E + 4\cdot 5Fx + \text{etc.} \qquad (5)$$

Differentiating (5), and dividing by 4 and dx, we have

$$m\frac{1}{(1 + x)^5} = 5F + \text{etc.}\ddagger \qquad (6)$$

We have now gone far enough to enable us to determine the coefficients A, B, C, D, E, and F, and these will probably reveal the law of the series.

As all the above equations are to be true for all values of x, and as the coefficients A, B, C, etc., are constant, *i. e.*, have the same values for one value of x as for another, if we can determine their values for one value of x, these will be their values in all cases. Now, making $x = 0$, we have, from (1), $A = \log 1 = 0$;

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* What the relation of the modulus to the base is, we are not now concerned to know; it will be determined hereafter.

† The number is $1 + x$; hence the differential is m times the differential of $1 + x$ divided by the number $1 + x$.

‡ Of course the student will observe what forms the succeeding terms in this and the other similar cases would have. Thus here we should have $5F + 5\cdot 6Gx + 3\cdot 5\cdot 7Hx^2 + \text{etc.}$

from (2), $B = m$; from (3), $C = -\frac{1}{2}m$; from (4), $D = \frac{1}{3}m$; from (5), $E = -\frac{1}{4}m$; from (6), $F = \frac{1}{5}m$. These values substituted in (1) give

$$\log(1+x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right),$$

the law of which is evident. This is the *Logarithmic Series,* and should be fixed in memory.

SCH.—The Napierian system of logarithms is characterized by the modulus being 1 ($m = 1$). Hence the *Napierian logarithmic series* is

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}$$

190. COR. 1.—*The logarithms of the same number in different systems are to each other as the moduli of those systems.*

This is evident from the general logarithmic series. Thus the logarithm of $1 + x$ in a system whose modulus is m, is expressed

$$\log_m(1+x)^* = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right):$$

and the logarithm of the same number in a system whose modulus is m' is expressed

$$\log_{m'}(1+x)^* = m'\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right).$$

Now, as the number $(1 + x)$ is, by hypothesis, the same in both cases, x is the same. Hence, dividing the members of the first by the corresponding members of the second, we have $\dfrac{\log_m(1+x)}{\log_{m'}(1+x)} = \dfrac{m}{m'}$.

191. COR. 2.—*Having the logarithm of a number in the Napierian system, we have but to multiply it by the modulus of any other system to obtain the logarithm of the same number in the latter system.*

Or, the logarithm of a number in any system divided by the logarithm of the same number in the Napierian system, gives the modulus of the former system.

192. Prob.—*To adapt the Napierian logarithmic series to numerical computation so that it can be conveniently used for computing the logarithms of numbers.*

SOL.—That $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}$, is not in a practicable form for computing the logarithms of numbers will be evident if we make the attempt. Thus, suppose we wish to compute the logarithm of 3. Making

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* The subscripts m and m' are used to distinguish between the systems, as $\log(1+x)$ is not the same in one system as in the other. Read $\log_m(1+x)$, "logarithm of $1+x$ in a system whose modulus is m," etc.

$x = 2$, we have $\log(1+2) = \log 3 = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \frac{2^5}{5}$ — etc., a series in which the terms are growing larger and larger (a diverging series).

We wish a series in which the terms will grow smaller as we extend it (a converging series). Then the farther we extend the series, the more nearly shall we approximate the logarithm sought. To obtain such a series, substitute $-x$ for x in the Napierian logarithmic series, and we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \text{etc.}$$

Subtracting this from the former series, we have

$$\log(1+x) - \log(1-x) = \log\left(\frac{1+x}{1-x}\right) = 2(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \text{ etc.}).$$

Now put $x = \frac{1}{2z+1}$, whence $1+x = 1 + \frac{1}{2z+1} = \frac{2z+2}{2z+1}$, $1-x = \frac{2z}{2z+1}$, and $\frac{1+x}{1-x} = \frac{1+z}{z}$. Hence, as $\log\left(\frac{1+z}{z}\right) = \log(1+z) - \log z$, substituting, and transposing,

$$\log(1+z) = \log z + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \frac{1}{7(2z+1)^7} + \text{etc.}\right). \quad \text{(A)}$$

This series converges quite rapidly, especially for large values of z, and is convenient for use in computing logarithms.

193. Prob.—*To compute the Napierian logarithms of the natural numbers* 1, 2, 3, 4, *etc., ad libitum.*

SOLUTION.—In the first place we remark that it is necessary to compute the logarithms of *prime* numbers only, since the logarithm of a composite number is equal to the sum of the logarithms of its factors (*178*).

Therefore beginning with 1, we know that $\log 1 = 0$ (*173*).

To compute the logarithm of 2, make $z=1$, in series (A), and we have $\log(1+1) - \log 1 = \log 2 = 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \frac{1}{9\cdot 3^9} + \frac{1}{11\cdot 3^{11}} + \frac{1}{13\cdot 3^{13}} + \frac{1}{15\cdot 3^{15}} + \text{etc.}\right)$.

The numerical operations are conveniently performed as follows:

3	2.00000000		
9	.66666667	1	.66666667*
9	.07407407	3	.02469136
9	.00823045	5	.00164609
9	.00091449	7	.00013064
9	.00010161	9	.00001129
9	.00001129	11	.00000103
9	.00000125	13	.00000009
	.00000014	15	.00000001

$\therefore \log 2 = .69314718$*

* Though the decimal part of a logarithm is generally not exact, it is not customary to annex the + sign.

Second. To find log 3, make $z = 2$, whence

$$\log 3 = \log 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9} + \text{etc.}\right).$$

Computation.				
	5	2.00000000		
	25	.40000000	1	.40000000
	25	.01600000	3	.00533333
	25	.00064000	5	.00012800
	25	.00002560	7	.00000366
		.00000102	9	.00000011
				.40546510
			log 2 =	.69314718
			∴ log 3 =	1.09861228

Third. To find log 4. Log 4 = 2 log 2 = 2 × .69314718 = 1.38629436

Fourth. To find log 5. Let $z = 4$, whence

$$\log 5 = \log 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7} + \text{etc.}\right).$$

Computation.				
	9	2.00000000		
	81	.22222222	1	.22222222
	81	.00274348	3	.00091449
	81	.00003387	5	.00000677
		.00000042	7	.00000006
				.22314354
			log 4 =	1.38629436
			∴ log 5 =	1.60943790

In like manner we may proceed to compute the logarithms of the *prime* numbers from the formula, and obtain those of the composite numbers on the principle that the logarithm of the product equals the sum of the logarithms of the factors.

Thus, the Napierian logarithm of the base of the common system, 10, = log 5 + log 2 = 2.30258508.

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194. *Prop.*—*The modulus of the common system is* .43429448+.

Dem.—Since the logarithm of a number, in any system, divided by the Napierian logarithm of the same number is equal to the modulus of that system (***191***), we have

$$\frac{\text{Com. log } 10}{\text{Nap. log } 10} = \text{modulus of common system.}$$

But com. log 10 = 1, and Nap. log 10 = 2.30258508, as found above. Hence,

$$\textit{Modulus of common system} = \frac{1}{2.30258508} = .43429448.$$

Tables of Logarithms.

195. As one of the most important uses of logarithms is to facilitate the performance of multiplication, division, involution, and evolution, when the numbers are large, according to (***178-181***), it is necessary to have at hand a table containing the logarithms of numbers. Such a table of common logarithms is usually found in treatises on trigonometry and on surveying, or in a separate volume of tables.* These tables usually contain the common logarithms of numbers from 1 to 10000, with provision for ascertaining therefrom the logarithms of other numbers with sufficient accuracy for practical purposes. Four pages of such a table will be found at the close of this volume.

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196. Prob.—*To find the logarithm of a number from the table.*

Solution.—The logarithm of any number from 1 to 100 inclusive can be taken directly from the first page of the table. Thus log 2 = 0.301030, and log 21 = 1.322219.†

To find the logarithm of any number from 100 to 999 inclusive, look for the number in the column headed N, and opposite the number in the first column at the right is the mantissa of the logarithm. The characteristic is known by (***185***). Thus log 182 = 2.260071; log 135 = 2.130334.

To find the logarithm of any number represented by 4 figures, find the first 3 left-hand figures in column N, and opposite this at the right in the column which has the fourth figure at its head, will be found the last four figures of the mantissa. The other two figures of the mantissa will be found in the 0 column, oppo-

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* Mathematicians and practical computers generally use more complete and extended tables than those found in connection with such elementary treatises. The common tables give five places of decimals in the mantissa. Those in connection with this series give six. Callet's tables edited by Hasler are standard eight-place logarithms. Vega's tables are among the best. Dr. Bremiker's edition, translated by Prof. Fischer, is a favorite. Köhler's edition of Vega's contains Gaussian logarithms. Vega's tables are seven-place. Ten-place logarithms are necessary for the more accurate astronomical calculations. Prof. J. Mills Peirce, of Harvard, has recently issued an elegant little folio edition of tables containing among other things a table of three-place logarithms which is very convenient for most uses.

† This page is really unnecessary, since nothing can be found from it which cannot be found with equal ease from the succeeding part of the table. Thus, the mantissa of log 2 is the same as the mantissa of log 200; and the mantissa of log 21 is the same as that of log 210.

site the first three figures of the number or just above, unless heavy dots have been passed or reached in running across the page to the right, in which case the first two figures of the mantissa will be found in the 0 column just *below* the number. The places of the heavy dots must be supplied with 0's. The characteristic is determined by (***185***). Thus log 1316=3.119256; log 2042=3.310056; log 1868 = 3.271377.

To find the logarithm of a number represented by more than 4 figures. Let it be required to find the logarithm of 1934261. Finding the mantissa corresponding to the first four figures (1934) as before, we find it to be .286456. Now in the same horizontal line and in the column marked D, we find 225, which is called the *Tabular Difference.* This is the difference between the logarithms of two consecutive numbers at this point in the table. Thus 225 (millionths) is the difference between the logarithms of 1934 and 1935, or, as we are using it, between the logarithms of 1934000 and 1935000, which differences are the same. Now, assuming that, if an increase of 1000 in the number makes an increase of 225 (millionths) in the logarithm, an increase of 261 in the number will make an increase of $\frac{261}{1000}$, or, .261, of 225 (millionths) in the logarithm,* we have .261 × 225 (millionths) = 59 (millionths), omitting lower orders, as the amount to be added to the logarithm of 1934000 to produce the logarithm of 1934261. Adding this and writing the characteristic (***185***) we have log 1934261 = 6.286515. In like manner the logarithm of any other number expressed by more than four figures may be found.

197. Sch.—As the *mantissa* of a mixed integral and decimal fractional number, or of a number entirely decimal fractional, is the same as that of an integral number expressed by the same figures (***184***), we can find the mantissa of the logarithm of such a number as if the number were wholly integral, and determine the characteristic by (***185***).

198. Prob.*—To find the number corresponding to a given logarithm.*

Solution.—Let it be required to find the number corresponding to the logarithm 4.234567. Looking in the table for the *next less* mantissa, we find .234517, the number corresponding to which is 1716 (no account being taken as to whether it is integral, fractional, or mixed; as in any case, the figures will be the same). Now, from the *tabular difference,* in column D, we find that an increase of 253 (millionths) upon this logarithm, would make an increase of 1 in the number, making it 1717. But the given logarithm is only 50 greater than the logarithm of 1716; hence, it is assumed (though only approximately correct) that the increase of the number is $\frac{50}{253}$ of 1, or .1976 +. This added (the figures annexed) to 1716, gives 17161976 +. The characteristic of the given logarithm being 4, the number lies between the 4th and 5th powers of 10, and hence has 5 integral places. ∴ 4.234567 = log 17161.976 +. In like manner the number corresponding to any logarithm can be found.

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* This assumption, though not strictly correct, is sufficiently accurate for all ordinary purposes.

199. *Prop.*—*The Napierian base is* 2.718281828.

DEM.—Let e represent the base of the Napierian system. Then by (***190***)

com. log e : Nap. log e : : .43429448 : 1.

But the logarithm of the base of a system, taken in that system is 1, since $a^1 = a$. Hence, Nap. log $e = 1$, and com. log $e = .43429448$. Now finding from a table of common logarithms the number corresponding to the logarithm .43429448, we have $e = 2.718281828$.

EXAMPLES.

1. If 3 were the base of a system of logarithms, what would be the logarithm of 81? Of 729? If 5 were the base, of what number would 3 be the logarithm? Of what 2? Of what 4?

2. If 2 were the base, what would be the logarithm of $\frac{1}{4}$? Of $\frac{1}{8}$? Of $\frac{1}{32}$?

3. If 16 were the base, of what number would .5 be the logarithm? Of what .25?

4. In the common system we find that log 156=2.193125. Show that this signifies that $10^{\frac{2193125}{1000000}}=156$.

5. Log 1955=3.291147. To what power does this indicate that 10 is to be raised, and what root extracted to make 1955?

6. Find from the table at the close of the volume what root of what power of 10 equals 2598.

7. Multiply 1482 by 136 by means of logarithms, using the table at the close of the volume. (See ***178.***)

8. Perform the following operations by means of logarithms: 1168×1879; $2769 \div 187$; 15.13×1.3476; $257.16 \div 18.5134$; $.126 \div 6.1413$; $.11257 \times .00126$; $(1278.6)^2$; $(112.37)^3$.

9. Perform the following operations by means of logarithms: $\sqrt{2}$ to 5 places of decimals; $\sqrt[3]{5}$ to 3 places of decimals; $\sqrt[3]{2341564273}$ to two places of decimals; $\sqrt{3015618}$ to 4 places of decimals.

10. Perform the following operations by means of logarithms: $\sqrt[3]{.01234}$ to 4 places of decimals; $\sqrt{.03125}$ to 5 places of decimals; $\sqrt[5]{.0002137}$ to 5 places of decimals.

SUG'S.—Log .01234=$\bar{2}.091315$. Now to divide this by 3, we have to remember that the characteristic *alone* is negative, *i. e.* that $\bar{2}.091315 = -2 + .091315$, or

-1.908685, which is all negative. Dividing this by 3, we have $-.636228$, or $0-.636228=\bar{1}.363772$. But a more convenient way to effect the division is to write $\bar{2}.091315=\bar{3}+1.091315$, and dividing the latter by three we obtain $\bar{1}.363772$, in which the characteristic *alone* is negative, thus conforming to the tables.

To divide $\overline{13}.341652$ by 4, we write for $\overline{13}.341652$, $-16+3.341652$, and dividing the latter obtain $\bar{4}.835413$.

11. Divide as above $\overline{11}.348256$ by 3; $\overline{17}.135421$ by 5; $\bar{1}.341263$ by 6.

12. Given the following to compute x by logarithms:

$201.56:134.201::18.654:x$; $\quad 2350.64:.212::1.1123:x$;

$x:234.008::15.738:200.56$; $\quad 123:x::2.01:.03$.

13. Having $y=\sqrt{\dfrac{a^2-x^2}{1+x}}$ to express the equivalent operations in logarithms.

SUG'S. $y=\sqrt{(a-x)(a+x)\div(1+x)}$. $\therefore \log y=\frac{1}{2}[\log(a-x)+\log(a+x)-\log(1+x)]$.

14. Given $y=x^{\frac{2}{3}}(1-x^2)^{\frac{1}{2}}$ to express the equivalent operations in logarithms. Also $y=\sqrt{\dfrac{ax}{by}}$. Also $y=\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$.

Also $y=\dfrac{\sqrt[3]{x-x^2}}{\sqrt{y^3}}$. Also $y=\sqrt[n]{\dfrac{a^mb^p}{c^t}}$. Also given $\dfrac{a^m}{b^n}:\dfrac{c^{\frac{1}{2}}}{d^{\frac{1}{t}}}::\sqrt[n]{m^2-x^2}:y$ to express $\log y$.

15. Differentiate $y=\log(a^2-x^2)$.

SUG'S.—Write $y=\log(a+x)+\log(a-x)$. Then differentiating, we have $dy=\dfrac{mdx}{a+x}-\dfrac{mdx}{a-x}$. Or differentiating without factoring, we have $dy=\dfrac{d(a^2-x^2)^*}{a^2-x^2}=-\dfrac{2mxdx}{a^2-x^2}$. When reduced the results are the same, but the former is usually the more elegant method.

16. Differentiate the following: $y=\log(1-x)$; $y=\log ax$; $y=\log x^3$; $y=\log\dfrac{a}{x}$; $y=\log\sqrt{1+x}$.

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* This form signifies that a^2-x^2 *is to be* differentiated. The operation is only indicated, not performed.

SUG'S.—Remember that $\log x^3 = 3 \log x$; and also that $\log \sqrt{1+x} = \frac{1}{2}\log(1+x)$.

17. Find from the table at the close of the volume that Nap. log 1564=7.3550018. Find in like manner the Napierian logarithms of 5, 120, and 2154372.

18. Knowing that the Napierian logarithm of 22 is 3.0910425, how would you find the common logarithm of 23 from the logarithmic series (***192***)?

19. The common logarithm of 25 is 1.39794. What is the modulus, and what the base of a system which makes the logarithm of 25 2.14285?

QUERY.—How do you see at a glance that the required base is a little less than 5?

SECTION V.

SUCCESSIVE DIFFERENTIATION, AND DIFFERENTIAL COEFFICIENTS.

200. Prop.—*Differentials, though infinitesimals, are not necessarily equal to each other.*

DEM.—Thus, let $y=2x^3$. Then $dy=6x^2dx$. Now, for all finite values of x, dy is an infinitesimal, since no finite number of times the infinitesimal dx can make a finite quantity, and dy is $6x^2$ times dx. But for $x=1$, dy is 6 times dx; for $x=2$, dy is 24 times dx; for $x=3$, dy is 54 times dx.

201. COR.—*When* $y=f(x)$, dy *is generally a variable, and hence can be differentiated as any other variable.*

202. NOTATION.—The differential of dy is written d^2y, and read "second differential of y." The differential of d^2y is written d^3y, and read "third differential of y," etc. The superiors 2 and 3 in such cases are not of the nature of exponents, as the d is not a symbol of number.

203. In differentiating $y=f(x)$ successively, it is customary to regard dx as constant. This is conceiving x to change (grow) by *equal infinitesimal increments*, and thence ascertaining how y varies. In general, y will not vary by equal increments when x does, as appears from the demonstration above.

204. A Second Differential is the difference between two consecutive states of a first differential.—***A Third Differential*** is the difference between two consecutive states of a second differential, etc.

ILL.—In the function $y=2x^3$, if x passes to the next state, we have $dy=6x^2dx$. Now dy, though an infinitesimal, is still a variable, for it is equal to $6dx$ times x^2, and x is a variable. Hence if x takes an infinitesimal increment, dy will pass to a consecutive state. In other words, we can differentiate $dy=6dx\,x^2$, just as we could $u=mx^2$, dy being a variable function, $6dx$ a constant factor, and x the variable. Representing the differential of dy by d^2y, we have $d^2y=6dx\cdot 2xdx$, or $d^2y=12xdx^2$, dx^2 being the square of dx, not the differential of x^2. To indicate the latter we would write $d(x^2)$.

EXAMPLES.

1. Given $y=3x^5-2x^2$ to find the third differential of y, or d^3y.

SOLUTION.—Differentiating $y=3x^5-2x^2$, we have $dy=15x^4dx-4xdx$. Now, regarding dx as constant, and differentiating again, we have $d^2y=60x^3dx^2-4dx^2$.* Differentiating again in like manner, we obtain $d^3y=180x^2dx^3$, the second term disappearing, since $4dx^2$ is constant.

2. Given $y=2x^3-3x+5$ to find the second differential of y, *i. e.* d^2y.

3. Given $y=(x-a)^3$ to find the third differential of y.

SUG'S. $dy=3(x-a)^2dx$, $d^2y=6(x-a)dx^2$, $d^3y=6dx^3$.

4. Given $y=Ax+Bx^2+Cx^3+Dx^4$, to find the 4th differential of y, A, B, C, and D, being constant. $d^4y=4\cdot 3\cdot 2\,Ddx^4$.

5. Differentiate $y=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+$ etc., 5 times in succession.

6. Differentiate $y=(x-1)(x-2)(x-3)(x-4)$ twice in succession without expanding.

SUG'S. $dy=(x-2)(x-3)(x-4)dx+(x-1)(x-3)(x-4)dx+(x-1)(x-2)(x-4)dx+(x-1)(x-2)(x-3)dx.$

$=[(x-2)(x-3)(x-4)+(x-1)(x-3)(x-4)+(x-1)(x-2)(x-4)+(x-1)(x-2)(x-3)]dx.$

$$d^2y=[(x-3)(x-4)dx+(x-2)(x-4)dx+(x-2)(x-3)dx+(x-3)(x-4)dx+(x-1)(x-4)dx+(x-1)(x-3)dx+(x-2)(x-4)dx+(x-1)(x-4)dx+(x-1)(x-2)dx+(x-2)(x-3)dx+(x-1)(x-3)dx+(x-1)(x-2)dx]dx.$$

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* To differentiate $15x^4dx$, calling dx constant, we may write $15dx\,x^4$. Now $15dx$ is constant. Hence differentiating x^4, we have $4x^3dx$, which multiplied by the constant $15dx$, gives, as above, $60x^3dx^2$. The dx^2 is "the square of dx," not the differential of x^2.

$= [(x-3)(x-4) + (x-2)(x-4) + (x-2)(x-3) + (x-3)(x-4) + (x-1)(x-4) + (x-1)(x-3) + (x-2)(x-4) + (x-1)(x-4) + (x-1)(x-2) + (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)]dx^2.$

7. As above, differentiate $y = (x - a)(x - b)(x - c)$ twice in succession without expanding.

DIFFERENTIAL COEFFICIENTS.

205. *The First Differential Coefficient* is the ratio of the differential of a function to the differential of its variable. Thus, if $y = f(x)$, and $dy = f'(x)dx$, $\frac{dy}{dx} = f'(x)$, and $\frac{dy}{dx}$, or its equivalent $f'(x)$, is the first differential coefficient of y, or $f(x)$.

ILL.—The meaning of this is simple. Thus, if $y = 2x^4$, $\frac{dy}{dx} = 8x^3$; that is, if x takes an infinitesimal increment dx, y takes an infinitesimal increment dy, which is to dx, as $8x^3$ is to 1, or the ratio of dy to dx is $8x^3$. In still other words, y increases $8x^3$ times as fast as x. The reason for calling this a differential coefficient, is that it is the *coefficient* by which the increment (dx) of the variable must be multiplied to give the increment (dy) of the function.

206. *The Second Differential Coefficient* is the ratio of the second differential of a function to the square of the differential of the variable. Thus, if $y = f(x)$, $dy = f'(x)dx$, and $d^2y = f''(x)dx^2$, $\frac{d^2y}{dx^2} = f''(x)$, $\frac{d^2y}{dx^2}$ or its equivalent $f''(x)$, is the second differential coefficient of y, or $f(x)$. In like manner *Third*, *Fourth*, etc., differential coefficients are the ratios respectively of the third, fourth, etc., differentials of a function, to the cube, fourth power, etc., of the differential of the variable. Thus, if $y = f(x)$, $dy = f'(x)dx$, $d^2y = f''(x)dx^2$, $d^3y = f'''(x)dx^3$, and $d^4y = f^{iv}(x)dx^4$, the successive differential coefficients are $\frac{dy}{dx} = f'(x)$, $\frac{d^2y}{dx^2} = f''(x)$, $\frac{d^3y}{dx^3} = f'''(x)$, and $\frac{d^4y}{dx^4} = f^{iv}(x)$.

ILL.—Too much pains cannot be taken by the student in order to get a clear conception of the meaning of the various symbols $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$, etc. To illustrate, suppose we have $y = 2x^4 - x^3 + 6$, whence $\frac{dy}{dx} = 8x^3 - 3x^2$, $\frac{d^2y}{dx^2}$*

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* To produce the successive differential coefficients we may produce the corresponding successive differentials as in the preceding examples, or we may proceed thus: $\frac{dy}{dx} = 8x^3 - 3x^2$ can be differentiated, remembering that dy is variable and dx constant, and it gives $\frac{d^2y}{dx} = 24x^2dx - 6xdx$, whence $\frac{d^2y}{dx^2} = 24x^2 - 6x$.

$=24x^2-6x$, $\frac{d^3y}{dx^3}=48x-6$, and $\frac{d^4y}{dx^4}=48$. Now in this case $y=f(x)$, *i. e.*, y is a function of x; so $\frac{dy}{dx}$ is also a function of x, being equal to $8x^3-3x^2$; but, as it is not *the same* function of x that y is, we call it the f prime function, and write $\frac{dy}{dx}=f'(x)$. In like manner $\frac{d^2y}{dx^2}=f''(x)$ means that $\frac{d^2y}{dx^2}$ is some function of x, *but a different one* from either y, or $\frac{dy}{dx}$. It may be observed that, in *this example*, $\frac{d^4y}{dx^4}$ is not a function of x, and hence the inquiry arises as to the propriety of the notation $\frac{d^4y}{dx^4}=f^{\text{iv}}(x)$. It must be remembered that this form of notation is the *general* form, and it is the *general* fact that $\frac{d^4y}{dx^4}$ *is* a function of x, though in *special cases* it may not be.

EXAMPLES.

1. Produce the 1st, 2d, 3d, and 4th differential coefficients of $y=x^5-3x^3+x-10$.

OPERATION. $dy=5x^4dx-9x^2dx+dx$, whence $\frac{dy}{dx}=5x^4-9x^2+1$. Differentiating the latter* $\frac{d^2y}{dx}=20x^3dx-18xdx$, whence $\frac{d^2y}{dx^2}=20x^3-18x$. Again differentiating, $\frac{d^3y}{dx^2}=(60x^2-18)dx$, whence $\frac{d^3y}{dx^3}=60x^2-18$. Finally, $\frac{d^4y}{dx^4}=120x$.

2. If $y=5x^2-3x$, what is the ratio of the increase of y to that of x, in general? What is it when $x=1$? When $x=2$? When $x=3$?

Ans. In general, y increases $10x-3$ times as fast as x. When $x=1$, y is increasing 7 times as fast as x. When $x=2$, y is increasing 17 times as fast as x.

3. If $y=x^5+2x^4-x+10$, what is the ratio of the 3d differential of y to the cube of the differential of x? What is it when $x=1$? When $x=\frac{1}{2}$? When $x=\frac{1}{3}$? What is the name of this ratio?

4. If $y=(a+x)^m$, what is the 1st differential coefficient of the function? What the 2d? What the 3d? What the 5th? What the 11th?

$$\frac{d^5y}{dx^5}=m(m-1)(m-2)(m-3)(m-4)(a+x)^{m-5}.$$

* See foot-note on preceding page.

5. Produce the first 5 successive differential coefficients of

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6.$$

207. Sch.—The successive differential coefficients of a function of the form $A+Bx+Cx^2+Dx^3+$ etc., or $x^n+Ax^{n-1}+Bx^{n-2}+$ etc., are readily written by inspection. Thus, call $x^4-2x^3+5x^2+x-12, f(x)$. Let $f'(x)$ mean the first differential coefficient, $f''(x)$ the second, $f'''(x)$ the third, etc. We have

$$f(x) = x^4 - 2x^3 + 5x^2 + x - 12.$$
$$f'(x) = 4x^3 - 6x^2 + 10x + 1.$$
$$f''(x) = 12x^2 - 12x + 10.$$
$$f'''(x) = 24x - 12.$$
$$f^{iv}(x) = 24.$$
$$f^{v}(x) = 0.$$ Here the processes terminate.

Each of the above is produced from the preceding by multiplying the coefficient of x in each term by the exponent of x in that term and diminishing the exponent by 1.

6. According to the method indicated in the last scholium, write out the successive differential coefficients of the function $2x^5+3x^4-5x^2+10$. Also of $2x^8-3x^{10}+x^{12}$. Also of $3+2x-4x^3+3x^5$.

SECTION VI.

TAYLOR'S FORMULA.

208. Def.—***Taylor's Formula*** is a formula for developing a function of the *sum* of two variables in terms of the ascending powers of one of the variables, and finite coefficients which depend upon the other variable, the form of the function, and its constants.

209. Def.—If $u = f(x+y)$, *i. e.*, if u is a function of the sum of the two variables x and y, and we differentiate as though one of the variables, as x or y, was constant, the differential coefficients thus formed are called ***Partial Differential Coefficients.*** The partial differential coefficients of u, when x is considered variable and y constant, are represented thus: $\frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}, \frac{d^4u}{dx^4}$, etc. When y is considered variable and x constant, we write the coefficients $\frac{du}{dy}, \frac{d^2u}{dy^2}, \frac{d^3u}{dy^3}, \frac{d^4u}{dy^4}$, etc.

210. Lemma.—*If* $u = f(x + y)$, *the partial differential coefficients* $\frac{du}{dx}$ *and* $\frac{du}{dy}$ *are equal.*

DEM.—Having $u = f(x+y)$, if x take an increment, we have $u + d_xu$* $= f(x + dx + y) = f[(x+y) + dx]$; whence $d_xu = f[(x+y) + dx] - f(x+y)$, since a differential is the difference between two consecutive states of the function. Again, if y take an increment, we have $u + d_yu = f(x + y + dy)$ $= f[(x+y) + dy]$; whence $d_yu = f[(x+y) + dy] - f(x+y)$. Now the *form* of the values of d_xu and d_yu, as regards the way in which x and y are involved, is the same; hence, if it were not for dx and dy, they would be absolutely equal. Passing to the differential coefficients by dividing the first by dx and the second by dy, we have $\frac{du}{dx} = \frac{f[(x+y)+dx]-f(x+y)}{dx}$, and $\frac{du}{dy} = \frac{f[(x+y)+dy]-f(x+y)}{dy}$. But, in differentiating, the differential of the variable enters into every term; hence $f[(x+y) + dx] - f(x+y)$, as it would appear in application, would have a dx in each term which would be cancelled by the dx in the denominator in the coefficient, and $\frac{du}{dx}$ would be independent of dx. In like manner $\frac{du}{dy}$ is independent of dy. Hence, finally, as these values of the partial differential coefficients are simply functions of $(x+y)$, of the same form, and not involving dx or dy, they are equal. Q. E. D.

ILL.—To make this clear, let $u = (x+y)^3$. Then $d_xu = 3(x+y)^2dx$, or $\frac{du}{dx} = 3(x+y)^2$. Again, $d_yu = 3(x+y)^2dy$, or $\frac{du}{dy} = 3(x+y)^2$. Hence we see that $\frac{du}{dx} = \frac{du}{dy}$. So, again, if $u = \log(x+y)$, $\frac{du}{dx} = \frac{1}{x+y}$, and $\frac{du}{dy} = \frac{1}{x+y}$; hence $\frac{du}{dx} = \frac{du}{dy}$.

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211. Prob.—*To produce Taylor's Formula.*

SOLUTION.—Let $u = f(x+y)$ be the function to be developed. It is proposed to discover the law of the development when the function can be developed in the form

$$u = f(x+y) = A + By + Cy^2 + Dy^3 + Ey^4 + \text{etc.}, \qquad (1)$$

in which A, B, C, etc., are independent of y, and dependent on x, the form of the function, and its constants.

Supposing x constant and differentiating with reference to y as variable, remembering that, as A, B, C, etc., are functions of x, and not of y, they will be considered constant, we have

$$\frac{du}{dy} = B + 2Cy + 3Dy^2 + 4Ey^3 + \text{etc.} \qquad (2)$$

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* As we are to consider the effect produced upon u by an increment in x, and also by an increment in y, we adopt a form of notation to distinguish between the increments of u. Thus d_xu means the increment which u takes in consequence of x having taken the increment dx, while y remained constant. So d_yu represents the increment of u consequent upon the increment dy of y.

Again, differentiating (1) with respect to x, y being supposed constant, and remembering that A, B, C, etc., are functions of x, we have

$$\frac{du}{dx} = \frac{dA^*}{dx} + \frac{dB^*}{dx}y + \frac{dC}{dx}y^2 + \frac{dD}{dx}y^3 + \frac{dE}{dx}y^4 + \text{etc.} \qquad (3)$$

Hence by (*210*)

$$B+2Cy+3Dy^2+4Ey^3, \text{etc.} = \frac{dA}{dx} + \frac{dB}{dx}y + \frac{dC}{dx}y^2 + \frac{dD}{dx}y^3 + \frac{dE}{dx}y^4 + \text{etc.} \quad (4)$$

Now, by the theory of indeterminate coefficients, the coefficients of the like powers of y are equal, and we have

$$B = \frac{dA}{dx}, \quad 2C = \frac{dB}{dx}, \quad 3D = \frac{dC}{dx}, \quad 4E = \frac{dD}{dx}, \quad \text{etc.}$$

But as (1) is true for all values of y, we may make $y=0$; whence $A = f(x) = u'$; letting u' represent the value of the function u, when $y=0$. Now, as A is independent of y, it will have the same value for one value of y as for another; hence $A = f(x) = u'$ is the general value of A.

Again, $B = \frac{dA}{dx}$. But as $A = u'$, a function of x, $dA = du'$, and $B = \frac{du'}{dx}$.

In like manner $2C = \frac{dB}{dx}$. But as $B = \frac{du'}{dx}$, $dB = d\left(\frac{du'}{dx}\right) = \frac{d^2u'}{dx}$, and $C = \frac{1}{2}\frac{d^2u'}{dx^2}$.

So, also, as $3D = \frac{dC}{dx}$, and $dC = \frac{1}{2}d\left(\frac{d^2u'}{dx^2}\right) = \frac{1}{2}\frac{d^3u'}{dx^2}$, $D = \frac{1}{\lfloor 3}\frac{d^3u'}{dx^3}$.

Similarly we find $E = \frac{1}{\lfloor 4}\frac{d^4u'}{dx^4}$, and the law of the series is apparent.

Finally, substituting the values of A, B, C, etc., in (1), we obtain

$$u = f(x+y) = u' + \frac{du'}{dx}\frac{y}{1} + \frac{d^2u'}{dx^2}\frac{y^2}{\lfloor 2} + \frac{d^3u'}{dx^3}\frac{y^3}{\lfloor 3} + \frac{d^4u'}{dx^4}\frac{y^4}{\lfloor 4} + \text{etc.}, \qquad (5)$$

which is Taylor's Formula.

212. SCH.—Taylor's Formula develops $u = f(x+y)$ into a series in which the *first term* is the value of the function when $y = 0$; the *second term* is the first differential coefficient of the function when $y = 0$, into y; the *third term* is the second differential coefficient of the function when $y = 0$, into $\frac{y^2}{\lfloor 2}$; etc., etc.

As u' is $f(x+y)$ when $y = 0$, we may write $f(x)$ for u', and for $\frac{du'}{dx}$, $f'(x)$; for $\frac{d^2u'}{dx^2}$, $f''(x)$; for $\frac{d^3u'}{dx^3}$, $f'''(x)$; etc., as before explained. The formula then becomes

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* These forms are indicated operations. Thus, as A is a function of x, when we differentiate with respect to x we write dA, and to pass to the differential coefficient have to divide by dx.

$$u=f(x+y)=f(x)+f'(x)\frac{y}{1}+f''(x)\frac{y^2}{\lfloor 2}+f'''(x)\frac{y^3}{\lfloor 3}+f^{\text{iv}}(x)\frac{y^4}{\lfloor 4}+\text{etc.} \quad (6)$$

This is a very important method of writing Taylor's Formula, and should be clearly understood, and firmly fixed in memory.

EXAMPLES.

1. Develop $(x+y)^5$ by Taylor's Formula.

SOLUTION.—Putting $u=(x+y)^5$, we have $u'=x^5$, $\frac{du'}{dx}=5x^4$, $\frac{d^2u'}{dx^2}=20x^3$, $\frac{d^3u'}{dx^3}=60x^2$, $\frac{d^4u'}{dx^4}=120x$, and $\frac{d^5u'}{dx^5}=120$. Here the coefficients terminate, as the differential of a constant is 0.

Substituting these values in (5) (*211*), or (6) (*212*), we have

$$u=(x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5.$$

The same as by the Binomial Formula.

2. Develop $(x-y)^7$ by Taylor's Formula, and compare the result with that obtained by means of the Binomial Formula. Also $(x+y)^{\frac{1}{2}}$. Also $(x-y)^{-2}$. Also $(x+y)^{-\frac{2}{3}}$.

3. Show that

$$u=\log(x+y)=\log x+\frac{y}{x}-\frac{y^2}{2x^2}+\frac{y^3}{3x^3}-\frac{y^4}{4x^4}+\text{etc.}$$

4. Develop $(x+y)^m$ by Taylor's Formula, thus deducing the Binomial Formula.

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213. Taylor's Formula is much used for developing a function of a single variable after the variable has taken an increment. When so used the increment may be conceived as finite or infinitesimal, only so that it be regarded as a variable.

Ex. 1. Given $y=2x^3-x^2+5x-11$, to find y', which represents the value of the function after x has taken the increment h.

SOLUTION.—In the function as given, we have $y=f(x)$, and are to develop $y'=f(x+h)$. By Taylor's Formula we have

$$y'=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+\frac{d^3y}{dx^3}\frac{h^3}{\lfloor 3}+\frac{d^4y}{dx^4}\frac{h^4}{\lfloor 4}+\text{etc.}$$

From $y=2x^3-x^2+5x-11$, we have $\frac{dy}{dx}=6x^2-2x+5$, $\frac{d^2y}{dx^2}=12x-2$,

$\frac{d^3y}{dx^3}=12$, and subsequent differential coefficients 0. Substituting these values in the formula, we obtain

$$y'=(2x^3-x^2+5x-11)+(6x^2-2x+5)h+(12x-2)\frac{h^2}{2}+(12)\frac{h^3}{2\cdot 3}$$
$$=2x^3-x^2+5x-11+(6x^2-2x+5)h+(6x-1)h^2+2h^3.$$

This result is easily verified by substituting $x+h$ for x in the value of y, as given in the example. Thus,

$$y'=2(x+h)^3-(x+h)^2+5(x+h)-11;$$

a result which will reduce to the same form as the other.

2. Given $y=3x^5-2x^2$, to develop y', the value of y when x takes the increment h.

SECTION VII.

INDETERMINATE EQUATIONS.

214. An Indeterminate Equation between two quantities, as x and y, is an equation which expresses the *only* relation which is required to exist between the two quantities.

Ill.—Suppose we have $2x+3y=7$, and that this is the *only* relation which is required to exist between x and y. Then is $2x+3y=7$ an indeterminate equation. So also, if $\frac{x}{a}-b=cy$ is the only relation required to exist between x and y, this is an indeterminate equation. In like manner $y^2=2x^3-3x$ is an indeterminate equation if it expresses the only relation which is required to exist between x and y.

The propriety of the term *indeterminate* is seen if we observe that such an equation does not fix the *values* of x and y, but only their *relation*. Thus, in the equation $2x+3y=7$, x may be 2, and y 1, and the equation be satisfied. So x may be 3, and y $\frac{1}{3}$, and the equation be satisfied. In fact, *any* value may be assigned to one of the quantities and a corresponding value found for the other. Hence the equation does *not determine* the values of the quantities.

215. An equation between three quantities is indeterminate if it expresses the only required relation between the quantities, or if there is but one other relation required to exist.

Ill.—Thus, if $2x+3y-5z=10$ is the only relation which is required to exist between x, y, and z, it is evident that the equation does not determine particular, definite values for x, y, and z. So also if, in addition to the relation expressed by this equation, it is required that $2x$ shall equal $6y$, or $2x=6y$, these two equations will not fix the values of x, y, and z. For if $2x=6y$, the former equation becomes $9y-5z=10$, which may be satisfied for any value of z, and a corresponding value of y, as shown above.

216. In general, if there are n quantities involved in any number of equations less than n, and these are the only relations required to exist between the n quantities, the equations are indeterminate.

217. In indeterminate equations the quantities between which the relation or relations are expressed are properly *variables, i. e.*, they are capable of having any and all values.*

Ill.—Thus in the indeterminate equation $5y - 3x = 12$, *any* value may be assigned to x, and a corresponding value found for y; or *any* value may be assigned to y, and a corresponding value found for x.

218. There are, however, many classes of problems which give rise to equations which are called indeterminate, although they are not absolutely so: in such problems there is some other condition imposed than the one expressed by the equation, but which condition is not of such a character as to give rise to an independent, simultaneous equation. Such an equation may have a number of values for the variables, or unknown quantities, involved, but not an unlimited number.

Ill.—Let it be required to find the *positive, integral* values of x and y which will satisfy the equation $2x + 3y = 35$. Now, if $2x + 3y = 35$ were the *only* relation required to exist between x and y, there would be an infinite number of values of each which would satisfy the equation, as shown above. But there is the added condition that x and y shall be positive integers. This greatly restricts the number of values, but does not furnish another equation between x and y. We may usually solve such a problem by simple inspection. Thus, in this case, we have $y = \frac{35 - 2x}{3}$. Now, trying the integral values of x till $2x$ becomes greater than 35, *i. e.* till $x = 18$, we can determine what integral values of x give positive integral values for y. For $x = 1$, $y = 11$. For $x = 2$, $y = 10\frac{1}{3}$; hence $x = 2$ is to be rejected. For $x = 3$, $y = 9\frac{2}{3}$, and $x = 3$ is to be rejected. For $x = 4$, $y = 9$; hence $x = 4$ and $y = 9$ are admissible, etc.

[Note.—This subject is not of sufficient importance to justify our going into a general discussion of it. We shall content ourselves with a few practical examples concerning simple indeterminate equations between two or three quantities, and these restricted to positive *integral* solutions. *The chief thing of importance is that the student comprehend the nature of an indeterminate equation.*]

* This statement requires us to include imaginary values.

EXAMPLES.

1. What positive, integral values of x and y will satisfy the equation $5x+7y=29$?

SOLUTION.—We may write $x=\frac{29-7y}{5}=5-y+\frac{4-2y}{5}=5-y+\frac{2(2-y)}{5}$. Now to make x positive we must have $7y<29$; and as y is to be an integer it can only have values less than 5. Again, to render x integral $\frac{2-y}{5}$ must be integral, or 0. Finally, as no value for y less than 5 will render $\frac{2-y}{5}$ integral or 0, except $y=2$, this is the only value of y which fulfills the conditions. Hence the answer is $y=2$, $x=3$.

2. What positive, integral values of x and y will satisfy the equation $11x-17y=5$?

SOLUTION.—We have $x=\frac{17y+5}{11}=y+\frac{6y+5}{11}$. From this we see that any positive value of y which will render $\frac{6y+5}{11}$ integral, will meet the conditions. Put $\frac{6y+5}{11}=m$ (an integer); whence $y=\frac{11m-5}{6}=m+5\,\frac{m-1}{6}$. To make this value of y integral $\frac{m-1}{6}$ must be integral. Put $\frac{m-1}{6}=s$ (an integer); whence $m=6s+1$. Now *any* positive integral value for s will fulfill the conditions. Thus, put $s=0$;* whence $m=1$, $y=1$, and $x=2$. Again, put $s=1$; whence $m=7$, $y=12$, and $x=19$. For $s=2$, $m=13$, $y=23$, and $x=36$, etc. Hence there is an infinite number of positive, integral values of x and y which satisfy the equation.

3. What positive, integral values of x and y will satisfy the equation $21x+17y=2000$?

SUG'S. $x=\frac{2000-17y}{21}$. $\therefore$ y is <118. Again $x=\frac{2000-17y}{21}=95+\frac{5-17y}{21}$, and $\frac{5-17y}{21}=m$. $\therefore$ m is negative, and $y=-m+\frac{5-4m}{17}$. Whence $\frac{5-4m}{17}=s$, and $m=\frac{5-17s}{4}=1-4s+\frac{1-s}{4}$. $\therefore$ s is +, and any value of s which renders $\frac{1-s}{4}$, 0 or integral, and gives $y<118$, will meet the conditions.

$s=1$, gives $m=-3$, $y=4$ and $x=92$.
$s=5$, " $m=-20$, $y=25$ and $x=75$.
$s=9$, " $m=-37$, $y=46$ and $x=58$.
$s=13$, " $m=-54$, $y=67$ and $x=41$.
$s=17$, " $m=-71$, $y=88$ and $x=24$.
$s=21$, " $m=-88$, $y=109$ and $x=7$.

---

* 0 is considered an integer.

Since any greater value of s makes $y > 118$, these are all the values of x and y which fulfill the conditions.

4. Find the positive, integral values of x and y which satisfy the following:

(a) $5x + 11y = 254$; (b) $7x + 13y = 71$;
(c) $9x + 13y = 2000$; (d) $17x = 542 - 11y$;
(e) $11x + 35y = 500$; (f) $19x - 117y = 11$;
(g) $117x - 128y = 95$; (h) $39x + 29y = 650$;
(i) $5x \pm 9y = 40$; (k) $5x \pm 9y = 37$.

Applications.

1. In how many ways can I pay a debt of $2 with 3-cent and 5-cent pieces?

Sug's.—Let $x =$ the number of 3-cent pieces and $y =$ the number of 5-cent pieces required. Then we are to determine in how many ways the equation $3x+5y=200$ can be satisfied for positive, integral values of x and y.

We find it to be in 13 ways, as follows:

$y =$ 1	4	7	10	13	16	19	22	25	28	31	34	37
$x =$ 65	60	55	50	45	40	35	30	25	20	15	10	5

This means that 1 5-cent piece and 65 3-cent pieces will pay the debt, or 4 5-cent and 60 3-cent, or 7 5-cent and 55 3-cent, etc.

2. A man hands his grocer $5 and tells him to put up the worth of it in 11-cent and 3-cent sugars. Can the grocer do it in even pounds? If so, in how many ways? What is the greatest number of pounds of the poorer sugar that he can use? What the least?

3. In how many ways can a debt of £50 be discharged with guineas and 3-shilling pieces? *Ans.*, Not at all.

4. If my creditor has only 3-shilling pieces and I only guineas, can he so make change with me that I can pay him £50? Can I pay him £201? In how many different ways? What is the least number of guineas and 3-shilling pieces? How is it if I have crowns instead of guineas? How if I have guineas and my creditor crowns? How if I have crowns and my creditor pounds?

5. In how many ways can a debt of £1000 be paid in crowns and guineas?

Sug.—Having obtained a few of the possible values of x and y, the law will become evident.

219. Indeterminate Equations between Three Quantities.

1. What are the positive integral values of x, y, and z which satisfy $3x+5y+7z=100$?

Solution.—We have $x=\frac{100-5y-7z}{3}$; whence as 1 is the least value that y or z can have, x cannot be greater than 29. Also $y=\frac{100-3x-7z}{5}$; whence y cannot be greater than 18. Also $z=\frac{100-5y-3x}{7}$; whence z cannot be greater than 14.*

Write $x=\frac{100-5y-7z}{3}=33-y-2z+\frac{1-2y-z}{3}$. Hence $\frac{1-2y-z}{3}$ must be an integer. Put $\frac{1-2y-z}{3}=m$; whence $y=-m+\frac{1-z-m}{2}$. From this we see that m is negative.

Let us now proceed to examine in succession for $z=1$, $z=2$, $z=3$, etc.

For $z=1$.—For this value of z, $x=31-y-\frac{2y}{3}$, and $y=-m-\frac{m}{2}$. From the latter we see that m must be an even negative number; and from the former, that y must be a multiple of 3. Hence the following computation:

For $m=0$, $y=0$, which is inadmissible.
For $m=-2$, $y=3$, and $x=26$.
For $m=-4$, $y=6$, and $x=21$.
For $m=-6$, $y=9$, and $x=16$.
For $m=-8$, $y=12$, and $x=11$.
For $m=-10$, $y=15$, and $x=6$.
For $m=-12$, $y=18$, and $x=1$.

Since the values of x decrease as m increases numerically, and 1 is the least admissible value of x, we have all the values of y and x which correspond to $z=1$.

For $z=2$.—For this value of z, $x=28-y+\frac{2(1-y)}{3}$, and $y=-m-\frac{1+m}{2}$. From the latter we see that m must be a negative *odd* number; and from the former, that y must be 1, or a unit more than a multiple of 3. Hence the following computation:

For $m=-1$, $y=1$, and $x=27$.
For $m=-3$, $y=4$, and $x=22$.
For $m=-5$, $y=7$, and $x=17$.
For $m=-7$, $y=10$, and $x=12$.
For $m=-9$, $y=13$, and $x=7$.
For $m=-11$, $y=16$, and $x=2$.

Hence, these are all the values of y and x which correspond to $z=2$.

* Of course the quantities need not come up to these limits.

The other values are as follows:

$$z=3\left\{\begin{array}{l|r|r|r|r|}y=2&5&8&11&14\\x=23&18&13&8&3\end{array}\right.; \qquad z=4\left\{\begin{array}{l|r|r|r|}y=3&6&9&12\\x=19&14&9&4\end{array}\right.;$$

$$z=5\left\{\begin{array}{l|r|r|r|}y=1&4&7&10\\x=20&15&10&5\end{array}\right.; \qquad z=6\left\{\begin{array}{l|r|r|r|}y=2&5&8&11\\x=16&11&6&1\end{array}\right.;$$

$$z=7\left\{\begin{array}{l|r|r|}y=3&6&9\\x=12&7&2\end{array}\right.; \qquad z=8\left\{\begin{array}{l|r|r|}y=1&4&7\\x=13&8&3\end{array}\right.;$$

$$z=9\left\{\begin{array}{l|r|}y=2&5\\x=9&4\end{array}\right.; \qquad z=10\left\{\begin{array}{l|}y=3\\x=5\end{array}\right.;$$

$$z=11\left\{\begin{array}{l|r|}y=1&4\\x=6&1\end{array}\right.; \qquad z=12\left\{\begin{array}{l|}y=2\\x=2\end{array}\right..$$

2. What positive, integral values of x, y, and z satisfy $17x + 19y + 21z = 400$?

SUG.—There are 10 sets of values.

3. What positive, integral values of x, y, and z satisfy $5x + 7y + 11z = 224$?

4. What positive, integral values of x, y, and z satisfy $6x + 8y + 5z = 12$? Also $2x + 3y + 5z = 41$?

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220. If the conditions of a problem furnish less equations than unknown quantities, the problem is *indeterminate*, and in general can have an infinite number of solutions. But if the solution be limited to positive, integral values, it can be effected as above. Thus, if there are two equations and three unknown quantities, one of the unknown quantities can be eliminated and the resulting equation solved as heretofore. In like manner if there are three equations and four unknown quantities, a single equation between two may be found and solved; or if four unknown quantities and but two equations, a single equation between three unknown quantities may be found and solved.

EXAMPLES.

1. Given $2x + 5y + 3z = 51$, and $10x + 3y + 2z = 120$, to find all the positive, integral values of x, y, and z.

2. Given $3x + 5y + 7z = 560$, and $9x + 25y + 49z = 2920$, to find all the positive, integral values of x, y, and z.

3. Given $2x + 11y - 3z = 10$, and $3x - 2y + 3z = 30$, to find all the positive, integral values of x, y, and z.

Applications.

1. I wish to expend \$100 in the purchase of three grades of sheep, worth respectively \$3, \$7, and \$17 per head. How many of each kind can I buy? In how many different ways can I make the purchase? How many of the first two kinds must I take in order to get the least possible number of the third kind?

2. A merchant has three kinds of goods. The value of 20 yards of the first, less the value of 21 yards of the second, is \$38; while the value of 3 yards of the second and 4 yards of the third is \$34. What is the price per yard of each kind, the question being restricted to even dollars? What if the latter restriction be removed?

3. In how many ways can I pay a debt of \$171 with \$20, \$15, and \$6 notes? What is the least number of \$20 notes that I can use? Of \$15 notes? What the greatest number of \$6 notes?

4. A farmer has calves worth \$10, \$11, and \$13 per head. What relative number of each must he take and sell them at the uniform rate of \$12, without gain or loss? If he is to sell only 15 animals, how must he select them?

5. A man bought 124 head of cattle, viz., pigs, goats, and sheep, for \$400. Each pig cost \$4½, each goat \$3⅛, and each sheep \$1¼. How many were there of each kind?

6. A grocer has an order for 150 pounds of tea at 90 cents per pound, but having none at that price, he would mix some at 75 cents, some at 87½ cents, and some at \$1.00 per pound. How much of each sort must he take?

Sug's.—The nature of the 4th and 5th problems restricts their solutions to positive integers. The 6th is, however, only restricted by its nature to positive numbers; they may be fractional as well as integral.

[See Complete School Algebra, subject *Alligation.*]

7. What quantity of raisins, at 10 cents, 18 cents, and 20 cents per pound, must be mixed together to fill a cask containing 150 pounds, and to be worth 19 cents a pound?

8. A wheel in 36 revolutions passes over 29 yards; and in x of these revolutions it describes z yds., y ft., and 5 in. What are the values of x, y, and z?

CHAPTER II.

LOCI OF EQUATIONS.

[NOTE.—This subject, though properly geometrical, is introduced here for the purpose of the elegant and clear illustrations which it affords of the abstract principles of the subject of Higher Equations. It is thought that the aid which it will afford the pupil in comprehending the principles of the succeeding chapter will more than compensate for the time required to master this. Moreover, the subject of Loci of Equations is of prime importance in a mathematical course, and is always pursued with pleasure by the pupil. No geometrical knowledge is required in reading this chapter, farther than the ability to draw and measure straight lines.]

221. Prop.—*Every equation between two variables,* having real roots,† may be interpreted as representing some line either straight or curved.*

This proposition will be made sufficiently evident for our present purpose, if we show how such equations can be made to represent lines. This we shall do by means of particular examples.

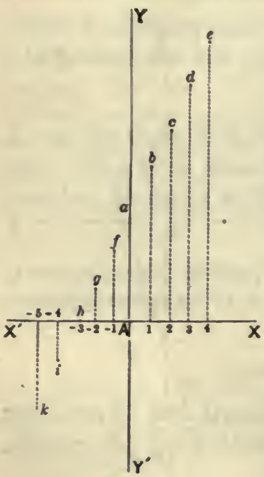

FIG. 1.

EXAMPLES.

1. Draw the line represented by the equation $y = 2x + 6$.

SOLUTION.—First, in all cases, draw two straight lines, as X′X and YY′, at right angles to each other, as in the figure. Then, in the equation $y = 2x + 6$, assign values (arbitrarily) to x, and find the corresponding values of y. Thus,

If $x=0$,	$y=6$,	Also, if $x=-1$,	$y=4$,
" $x=1$,	$y=8$,	" $x=-2$,	$y=2$,
" $x=2$,	$y=10$,	" $x=-3$,	$y=0$,
" $x=3$,	$y=12$,	" $x=-4$,	$y=-2$,
" $x=4$,	$y=14$,	" $x=-5$,	$y=-4$,
etc.,	etc.	etc.,	etc.

Having computed a few corresponding values of x and y in this way, we proceed with the figure, as follows: Measure off a distance A1 *to the right*

* This means simply, "having two variables, and only two, in it."

† The geometrical interpretation of imaginary loci does not come within our present purpose.

of A, of some convenient length, and call it the unit of distances. Draw $b\,1$, at 1, perpendicular to AX, and make it 8 units long (*i. e.*, 8 times as long as A 1). Now, b is at a distance 1 to the right of the line YY′, and 8 *above* the line X′X, and is hence a point in the line which our equation represents. In like manner, find the point c, 2 to the right of YY′, and 10 above X′X; and c is another point in the line represented by our equation. Again, when $x = 3$, $y = 12$. Hence, lay off three units to the right, as to 3 in the figure, and draw $d\,3$ perpendicular to X′X and 12 in length. Then is d another point in the line we seek. When $x = 4, y = 14$. Hence e is a point in the line; since it is 4 from YY′, and 14 from X′X. When $x = 0, y = 6$; whence a is a point in the line, as it is 0 distance from YY′, and 6 from X′X.

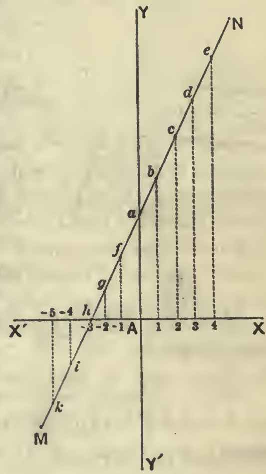

FIG. 2.

For negative values of x, we have, when $x = -1$, $y = 4$. Now, laying off negative values of x to the *left* from A, since we laid off positive values to the right, we measure from A to −1, the unit's distance, take $f\,1$ equal to 4 units, and thus find the point f. When $x = -2, y = 2$, and g is the corresponding point. When $x = -3, y = 0$; whence h is a point in the line, as it is 3 to the left of YY′ and 0 above X′X. When $x = -4$, $y = -2$. As this value of y is negative, we lay it off *below* X′X. Thus, taking from A to −4, a distance of 4 units, and from −4 to e, a distance of 2 units, i is a point in the line. Thus also k is a point in the line, since when $x = -5$, $y = -4$, and k is taken 5 to the left of YY′ and 4 below X′X.

This process might be continued indefinitely, both for positive and negative values of x. We might also use fractional values of x, as $x = \frac{1}{3}$, $x = \frac{1}{2}$, $x = 2\frac{1}{2}$, etc., and, finding the corresponding values of y, locate points between those found by taking integral values.

Finally, joining the points $e, d, c, b, a, f, g, h, i, k$, we have the line MN, which is represented by the equation $y = 2x + 6$. This line does not stop at M and N, of course, since we might produce it indefinitely either way, by continuing to take larger and larger values of x (numerically). In this case it is easy to see that the line is an indefinite straight line.

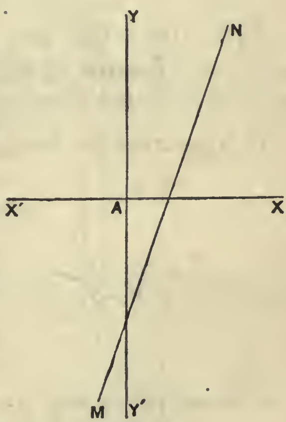

FIG. 3.

2. What line is represented by the equation $y = 3x - 6$? (See *Fig.* 3.)

SUG'S.—First compute a table of corresponding values of x and y, as in the preceding example; and then locate the points thus designated.

3. What line is represented by the equation $y = -2x + 4$? (See *Fig.* 4.)

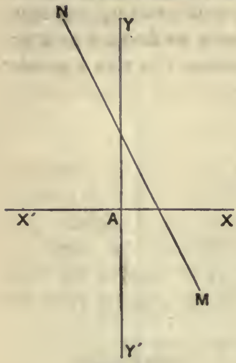

FIG. 4.

222. DEFINITIONS.—The assumed fixed lines X′X and YY′ are called the ***Axes of Reference***, or simply the ***Axes.*** A is called the ***Origin.*** X′X is the ***Axis of Abscissas***, and YY′ the ***Axis of Ordinates.*** The distance of a point from the axis of abscissas is called the ***Ordinate*** of the point; and the distance of a point from the axis of ordinates is called its ***Abscissa.*** The ordinate and abscissa of a point taken together are called its ***Co-ordinates.***

Abscissas measured to the right from the axis of ordinates are +, and those to the left −. Ordinates measured above the axis of abscissas are +, and those below −.

4 to 13. Draw as above the lines represented by the following equations: $y = x + 5$; $y = x - 5$; $y = -x + 5$; $y = -x - 5$; $y = 4x + 6$; $y = 4x - 6$; $y = -4x + 6$; $y = -4x - 6$; $2x - 3y = -5$; $\frac{x + 2y}{3} = 2$.

SUG.—Put such equations as the last two into the same form as the others before proceeding with the solution as above.

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223. DEF.—The line which is represented by an equation is called the ***Locus*** of the equation; and drawing the line in the manner indicated, is called *Constructing the Locus* of the equation.

14. Construct the locus of the equation $y = \frac{x}{1 + x^2}$.

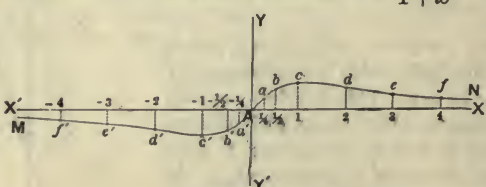

FIG. 5.

SOLUTION.—For $x = 0$,	$y = 0$,	For $x = -\frac{1}{2}$,	$y = -\frac{2}{5}$,
" $x = \frac{1}{4}$,	$y = \frac{4}{17}$,	" $x = -1$,	$y = -\frac{1}{2}$,
" $x = \frac{1}{2}$,	$y = \frac{2}{5}$,	" $x = -2$,	$y = -\frac{2}{5}$,
" $x = 1$,	$y = \frac{1}{2}$,	" $x = -3$,	$y = -\frac{3}{10}$,
" $x = 2$,	$y = \frac{2}{5}$,	" $x = -4$,	$y = -\frac{4}{17}$,
" $x = 3$,	$y = \frac{3}{10}$,	etc.	etc.
" $x = 4$,	$y = \frac{4}{17}$,		
etc.	etc.		

Now, laying off on the axis of abscissas to the right distances equal to $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 3, and 4, on some convenient scale, and at these points erecting ordinates equal respectively to $\frac{4}{17}$, $\frac{2}{5}$, $\frac{1}{2}$, $\frac{2}{5}$, $\frac{3}{10}$, and $\frac{4}{17}$ of the same scale, we find the points a, b, c, d, e, and f of the locus. We also see that if we continued to give x greater and greater values, y would continually grow less, but would only become 0 when $x=\infty$, for then we should have $y=\frac{x}{1+x^2}=\frac{x}{x^2}$* $=\frac{1}{x}=\frac{1}{\infty}=0$.

In like manner laying off the negative values of x, and the corresponding values of y, we find the points a', b', c', d', e', and f', and also find that y diminishes *numerically* as x increases numerically, and that for x negative y is always negative, and only becomes 0 when $x=-\infty$. Hence, the curve approaches the axis of abscissas to the left from below, as it does to the right from above, reaching it in either direction only at an infinite distance from the origin.

A line sketched through the points found represents the locus sought.

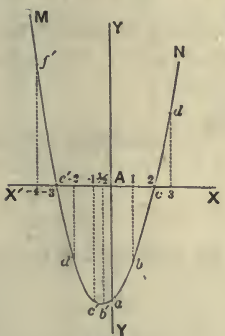

Fig. 6.

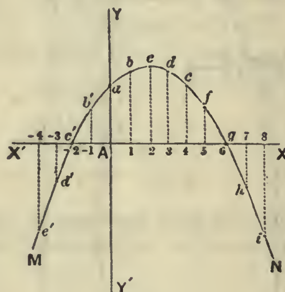

Fig. 7.

15 to 18. Construct the loci of the following equations: $y=x^2+x-6$ † (see *Fig.* 6); $y=3+x-\frac{1}{4}x^2$ (see *Fig.* 7); $y=x^2-4x+4$ (see *Fig.* 8); $y=x^2-3x+5$ (see *Fig.* 9).

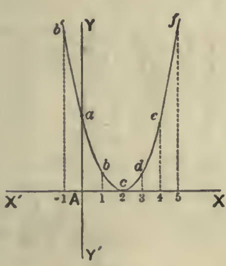

Fig. 8.

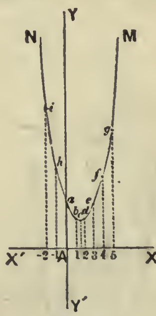

Fig. 9.

* Dropping the finite quantity 1, as producing no effect when added to the infinite x^2.

† In determining points in the locus, it is often necessary to attribute fractional values to x. Thus, in this case, to sketch the curve from a to c', we need an intermediate point. If there is any doubt about the character of the curve between two points, resolve the doubt in this way.

19 to 23. Construct the loci of the following equations: $y=x^3-\frac{7}{2}x^2+2x+2$ (see *Fig.* 10); $y=x^3-6x^2+13x-10$ (see *Fig.* 11); $y=x^3-2x-5$ (see *Fig.* 12); $y=x^2(5-x)$ (see *Fig.* 13); and $y=x^3-6x^2+11x-6$ (see *Fig.* 14).

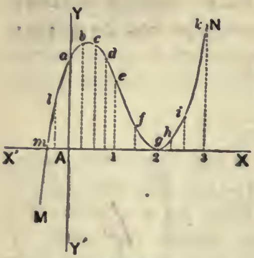

FIG. 10.

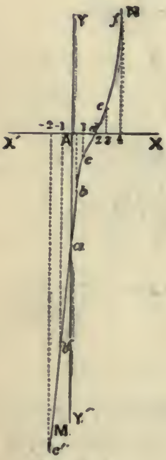

FIG. 11.

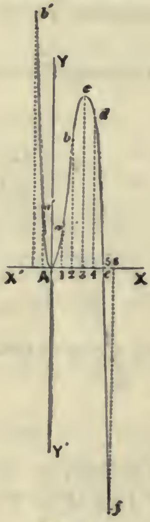

FIG. 13.

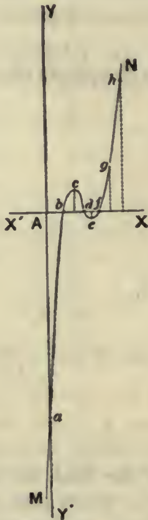

FIG. 14.

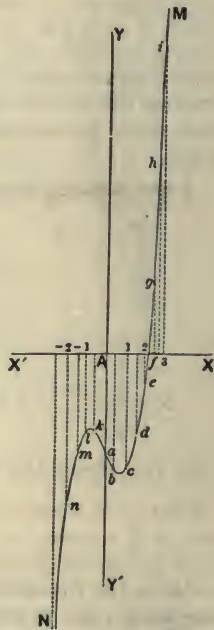

FIG. 12.

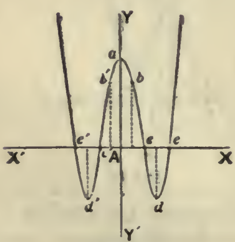

FIG. 15.

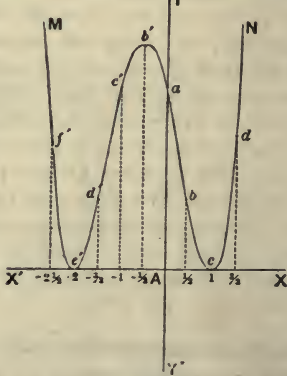

FIG. 16.

24 to 28. Construct the loci of the following equations: $y=x^4-5x^2+4$ (see *Fig.* 15); $y=x^4+2x^3-3x^2-4x+4$ (see *Fig.* 16); $y=x^4-9x^2+4x+12$ (see *Fig.* 17); $y=x^4-2x^3-7x^2-8x+16$ (see *Fig.* 18); and $y=x^4+x^3+x^2+x+1$ (see *Fig.* 19).

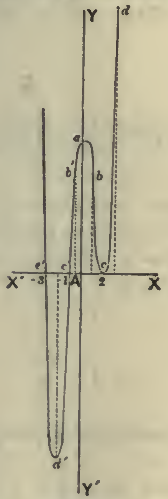

FIG. 17.

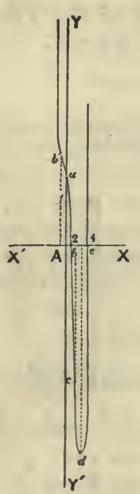

FIG. 18.

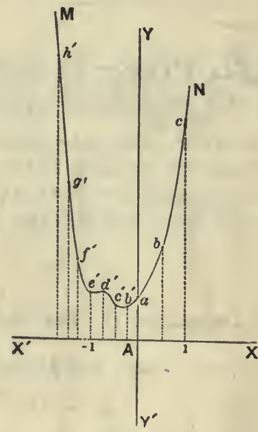

FIG. 19.

29. Construct the locus of the equation $y=x^5+4x^4-14x^2-17x-6$.

SUG'S. — In attempting to construct this locus, it is necessary to give x values from -3 to $+2$, including these values, and also to observe the character of the locus beyond these limits. But it will be found that for some values of x between these limits, y is inconveniently large. In sketching the figure, we may use one scale for laying off values of x, and another for laying off values of y. Thus in the figure given, the unit used for x is 6 times as great as that used for y. This is equivalent to constructing the locus $6y=x^5+4x^4-14x^2-17x-6$, or $y=\frac{1}{6}x^5+\frac{2}{3}x^4-\frac{7}{3}x^2-\frac{17}{6}x-1$. This locus has all the peculiarities of the one required (that is, all the turns, flexures, or bends), but is not of the same proportions. The portion represented is 6 times as wide in relation to its length as the required locus would have been.

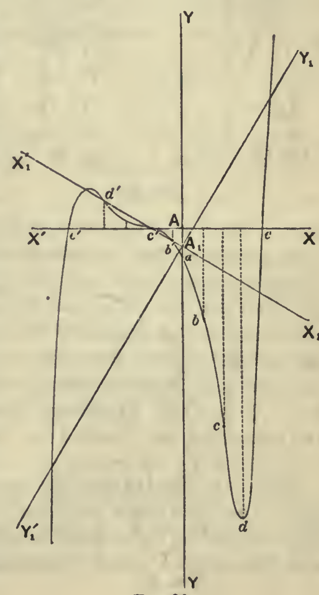

FIG. 20.

30 to 41. Construct the loci of the following equations, using a smaller scale for y than for x, as explained in

the suggestions above, when more convenient: $y = x^2 - 2x - 15$; $y = x^2 + 2x + 2$; $y = x^2 - 10x + 25$; $y = x^3 - 3x^2 - 8x - 10$; $y = 2x^3 - 12x^2 + 13x - 15$; $y = x^3 - x^2 - 8x + 12$; $y = x^3 - 2x^2 - 25x + 50$; $y = x^4 - 2x^3 + 8x - 16$; $y = x^4 + 2x^3 - 3x^2 - 4x + 4$; $y = x^4 - 6x^3 + 5x^2 + 2x - 10$; $y = x^5 + 5x^4 + x^3 - 16x^2 - 20x - 16$; and $y = 5x^5 - 4x^4 + 3x^3 - 3x^2 + 4x - 5$.

---

224. Prob.—*To construct the real roots of an equation containing only one unknown quantity.*

SOLUTION.—Put the equation in the form $f(x) = 0$, then write $y = f(x)$. Construct this equation, and the abscissas of the points where the locus cuts the axis of abscissas are the roots of the equation $f(x) = 0$. This is evident, since for these points, and for these only, $y = 0$, and we have $f(x) = 0$.

EXAMPLES.

1. Construct the real roots of the equation $x^2 - 3x - 2 = 0$.

SOLUTION.—We will first write $y = x^2 - 3x - 2$. Now, for $x = 0$, $y = -2$; for $x = 1$, $y = -4$; for $x = 2$, $y = -4$; for $x = 3$, $y = -2$; and for $x = 4$, $y = 2$. Hence we see that the locus of the equation $y = x^2 - 3x - 2$, cuts the axis of abscissas between $x = 3$, and $x = 4$, since it passes from below the axis of abscissas (where y is $-$) to above this axis (where y is $+$). There is therefore a root of $x^2 - 3x - 2 = 0$ between 3 and 4. To construct this root, we sketch the curve between $x = 3$ and $x = 4$, by finding the values of y for a few intermediate values of x, and then sketching the curve. Thus for $x = 3\frac{1}{2}$, $y = -\frac{1}{4}$; for $x = 3\frac{3}{4}$, $y = \frac{13}{16}$. Sketching the curve mn through these points, we find by measurement $Aa = 3.56$, as an approximate value of x in the equation $x^2 - 3x - 2 = 0$. (Verify by solving the equation.) To construct the other root, we notice that for $x = 0$, $y = -2$, and the curve cuts the axis of abscissas again at the left of the origin (probably, as it certainly does not cut it again at the right). Now, for $x = -1$, $y = 2$; whence we see that the locus cuts the axis between $x = 0$, and $x = -1$. For $x = -\frac{1}{2}$, $y = -\frac{1}{4}$; and for $x = -\frac{3}{4}$, $y = \frac{13}{16}$. Sketching the curve through these points, we have $m'n'$; and measuring Aa', we find the other value of x to be $-.56$.

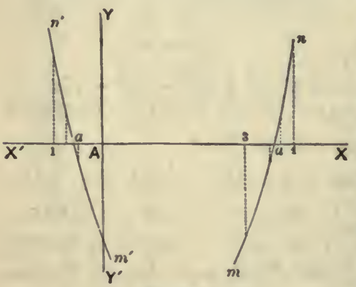

FIG. 21.

SUG.—For constructing the approximate roots in this manner, as we only need to sketch a small portion of the locus, in the vicinity of its intersection with the axis, we can use a much larger scale than would otherwise be practicable, and thus obtain a nearer approximation. With good instruments and some care, we can usually construct the root with tolerable accuracy to hundredths. When the locus cuts the axis quite obliquely, the approximation cannot be made as accurate.

2 to 7. As above, construct the real roots of the following: $x^2 - 8x = 14$; $x^3 - 12x^2 + 36x - 7 = 0$; $x^3 - x^2 - 10x + 6 = 0$; $x^3 - 7x + 7 = 0$; $x^4 - 12x^3 + 50x^2 - 84x + 49 = 0$; and $2x^5 - 7x^3 + 10x = 9$.

225. Sch.—This method of approximating the roots of equations geometrically is not given as a good practical method; but simply to assist the learner in comprehending some subsequent processes, and for its geometrical importance.

CHAPTER III.

HIGHER EQUATIONS.

SECTION I.

SOLUTION OF NUMERICAL HIGHER EQUATIONS HAVING COMMENSURABLE (OR RATIONAL) ROOTS.*

226. Equations of higher degrees than the second are called Higher Equations (***6–10,*** or same in Complete School Algebra). No general, practicable method of resolving such equations is known. Theoretical solutions of equations of the third and fourth degrees (cubics and biquadratics) are known; but these solutions are attended with practical difficulties in many cases, which render them nearly or quite useless. We are, however, able to obtain the *real* roots of *Numerical* Higher Equations, in *all* cases, either exactly, or to any required degree of approximate accuracy.

227. *The Real, Commensurable Roots* of numerical equations are usually found with little difficulty by the methods given in this section.

---

* A commensurable root (or number) is one which can be exactly expressed in the decimal notation, either in an integral, fractional, or mixed form. Thus, 4, $\frac{17}{3}$, $\sqrt{25}$, etc., are commensurable. But $\sqrt{21}$, $\sqrt[3]{17}$, etc., are incommensurable.

228. Prop.—*Every equation with one unknown quantity, having real and rational coefficients, can be transformed into an equation of the form*

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \quad - \ - \ - \ - \quad L = 0,$$

in which the exponents are all positive integers, the coefficient of x^n *is* 1, *and the coefficients of the other terms*, A, B, C, *etc., and also the absolute term* L, *are integers.*

DEM.—1st. If the unknown quantity occurs in any of the denominators in the given equation, remove it to the numerator by clearing of fractions. If there are then any negative exponents, multiply each term by the unknown quantity with a positive exponent equal to the numerically largest negative exponent. Then unite the terms with reference to the unknown quantity, and write them in order with the term containing the highest exponent, at the left, and so that the exponents shall diminish toward the right, transposing all the terms to the first member. The most complicated form which can then occur is

$$\frac{a}{b}y^{\frac{m}{n}} + \frac{c}{d}y^{\frac{r}{s}} + \frac{e}{f}y^t \quad - \ - \ - \ - \quad l = 0, \qquad (1)$$

in which any or all of the exponents may be fractions; and $\frac{m}{n} > \frac{r}{s} > t$, etc. is supposed.

2d. To free the equation of fractional exponents, substitute for the unknown quantity a new unknown quantity with an exponent equal to the least common multiple of the denominators of the exponents in the equation. Thus, in (1) put $y = z^{ns}$, whence $y^{\frac{m}{n}} = z^{ms}$, $y^{\frac{r}{s}} = z^{nr}$, and $y^t = z^{nst}$. These values substituted in the equation, will evidently give an equation of the form

$$\frac{a}{b}z^n + \frac{c}{d}z^{n-1} + \frac{e}{f}z^{n-2} \quad - \ - \ - \ - \quad l = 0, \qquad (2)$$

in which all that is essential concerning the exponents is that they should be all positive integers, decreasing in value from left to right, since in (1) $\frac{m}{n} > \frac{r}{s} > t$, etc.

3d. Now divide by the coefficient of z^n, and let the resulting equation be represented by

$$z^n + \frac{c'}{d'}z^{n-1} + \frac{e'}{f'}z^{n-2} \quad - \ - \ - \ - \quad l' = 0. \qquad (3)$$

Finally, put $z = \frac{x}{k}$, and substitute in (3), thus obtaining

$$\frac{x^n}{k^n} + \frac{c'}{d'}\frac{x^{n-1}}{k^{n-1}} + \frac{e'}{f'}\frac{x^{n-2}}{k^{n-2}} \quad - \ - \ - \ - \quad l' = 0. \qquad (4)$$

Multiplying (4) by k^n, and representing the absolute term by L, we have

$$x^n + \frac{c'k}{d'}x^{n-1} + \frac{e'k^2}{f'}x^{n-2} \quad . \ . \ . \ . \quad L = 0.$$

If now k be so taken that these numerators will be divisible by the denominators, and the quotients represented by A, B, C, etc., we have

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}\ \ .\ \ .\ \ .\ \ .\ \ L=0,$$

the form required.

EXAMPLES.

1. Transform $\dfrac{1}{x}+\frac{2}{3}x^{-5}+\frac{5}{6}x^{\frac{1}{2}}=2x^{\frac{2}{3}}+\frac{1}{2}x^{-\frac{5}{2}}+\dfrac{3}{x^2}-2$, into a form having positive integral exponents and coefficients, and having the coefficient of the highest power 1.

SOLUTION.—Multiplying by x^2, we have

$$x+\tfrac{2}{3}x^{-3}+\tfrac{5}{6}x^{\frac{5}{2}}=2x^{\frac{8}{3}}+\tfrac{1}{2}x^{-\frac{1}{2}}+3-2x^2. \qquad (1)$$

Multiplying (1) by x^3, we have

$$x^4+\tfrac{2}{3}+\tfrac{5}{6}x^{\frac{11}{2}}=2x^{\frac{17}{3}}+\tfrac{1}{2}x^{\frac{5}{2}}+3x^3-2x^5. \qquad (2)$$

Putting $x=y^6$, there results

$$y^{24}+\tfrac{2}{3}+\tfrac{5}{6}y^{33}=2y^{34}+\tfrac{1}{2}y^{15}+3y^{18}-2y^{30}.$$

Arranging with reference to the highest power of y,

$$2y^{34}-\tfrac{5}{6}y^{33}-2y^{30}-y^{24}+3y^{18}+\tfrac{1}{2}y^{15}-\tfrac{2}{3}=0, \text{ or}$$

$$y^{34}-\tfrac{5}{12}y^{33}-y^{30}-\tfrac{1}{2}y^{24}+\tfrac{3}{2}y^{18}+\tfrac{1}{4}y^{15}-\tfrac{1}{3}=0. \qquad (3)$$

Finally, put $y=\dfrac{z}{k}$, whence

$$\frac{z^{34}}{k^{34}}-\frac{5z^{33}}{12k^{33}}-\frac{z^{30}}{k^{30}}-\frac{z^{24}}{2k^{24}}+\frac{3z^{18}}{2k^{18}}+\frac{z^{15}}{4k^{15}}-\frac{1}{3}=0, \text{ or}$$

$$z^{34}-\frac{5k}{12}z^{33}-k^4z^{30}-\frac{k^{10}}{2}z^{24}+\frac{3k^{16}}{2}z^{18}+\frac{k^{19}}{4}z^{15}-\frac{k^{34}}{3}=0.$$

Now, if k be made 12, this equation will be of the required form.*

Notice that as $x=y^6$, and $y=\dfrac{z}{12}$, $x=\dfrac{z^6}{(12)^6}$; so that, if the value of z could be found, the value of x would be known by implication.

2. Show as above how to transform the following:

(a) $2y^{-\frac{17}{5}}+\frac{3}{4}y^{-\frac{1}{2}}+\dfrac{2}{y^3}-\frac{1}{5}y^{-\frac{7}{2}}=\dfrac{1}{y}+\frac{2}{5}y^2-\frac{1}{10}y^{-4}$;

(b) $\dfrac{2}{x}-3x+\frac{4}{5}x^{\frac{1}{2}}-1=1$;

(c) $\dfrac{x^2-1}{1+x^{\frac{1}{2}}}=1-x^{-2}$; (d) $\dfrac{1-x^{\frac{2}{3}}}{1+x^{-1}}=\dfrac{x^{-1}+3}{x+2}$;

(e) $\sqrt{1-x^2}=1-3x^{\frac{1}{3}}$; (f) $\sqrt{2x-3x^3}-x=\sqrt{1-x}$.

---

* This substitution would be tedious, and as it is our present purpose simply to show the *possibility* of the transformation, and the method of making it, the substitution is unnecessary.

229.—Since every equation with one unknown quantity, and real and rational coefficients, can be transformed into one of the form

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3} \; - - - - \; L=0, \qquad (1)$$

this will be taken as the typical numerical equation whose solution we shall seek in this and the succeeding sections; and we shall frequently represent it by $f(x)=0$, read "function x equals 0." The notation $f(x)$ signifies in general, as has been before explained, simply *any expression* involving x. Here we use it for this particular form of expression. We shall also use $f'(x)$ as the symbol for the first differential coefficient of this function.

---

230. *Prop.*—*When an equation is reduced to the form* $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} - - - - L = 0$, *the roots, with their signs changed, are factors of the absolute (known) term,* L.

DEM.—1st. The equation being in this form, if a is a root, the function is divisible by $x-a$. For, suppose upon trial $x-a$ goes into the polynomial $x^n + Ax^{n-1} +$, etc., Q times *with a remainder R.* (Q represents any series of terms which may arise from such a division, and R any remainder.) Now, since the quotient multiplied by the divisor + the remainder, equals the dividend, we have $(x-a)\,Q+R=x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3} - - - - L$. But this polynomial $= 0$. Hence $(x-a)\,Q+R=0$. Now, by hypothesis a is a root, and consequently $x-a=0$. Whence $R=0$, or there is no remainder.

2d. If now $x-a$ exactly divides $x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3} - - - - L$, a must exactly divide L, as readily appears from considering the process of division. Hence $-a$ is a factor of L, a being a root of the equation. Q. E. D.

231. COR. 1.—*If* a *is a root of* $f(x)=0$, $f(x)$ *is divisible by* $x-a$; *and, conversely, if* $f(x)$ *is divisible by* $x-a$, a *is a root of* $f(x)=0$.

DEM.—The first statement is demonstrated in the proposition, and the second is evident, since as $f(x)$ is divisible by $x-a$, let the quotient be $\varphi(x)$; whence $(x-a)\,\varphi(x)=0$. Now $x=a$ will satisfy this equation, since it renders $x-a=0$, and does not render $\varphi(x)$ infinity, since by hypothesis x does not occur in the denominator.*

---

232. *Prop.*—*If the coefficients and absolute term in* $x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3} - - - - L=0$, *are all integers, the equation can have no fractional root.*

* Could there be a term of the form $\frac{c}{x-a}$ in $\varphi(x)$, $x=a$ would render it ∞, and $(x-a)\varphi(x)$ would be $0 \times \infty$, which is indeterminate, since $0 \times \infty = 0 \times \frac{1}{0} = \frac{0}{0}$.

DEM.—Suppose in this equation $x = \frac{s}{t}$, $\frac{s}{t}$ being a simple fraction in its lowest terms. Substituting this value of x, we have

$$\frac{s^n}{t^n} + A\frac{s^{n-1}}{t^{n-1}} + B\frac{s^{n-2}}{t^{n-2}} + C\frac{s^{n-3}}{t^{n-3}} \cdots L = 0.$$

Multiplying by t^{n-1} we obtain

$$\frac{s^n}{t} + As^{n-1} + Bts^{n-2} + Ct^2s^{n-3} \cdots Lt^{n-1} = 0.$$

Now, by hypothesis, all the terms except the first are integral, and the first is a simple fraction in its lowest terms, as by hypothesis s and t are prime to each other. But the sum of a simple fraction in its lowest terms and a series of integers cannot be 0. Therefore x cannot equal $\frac{s}{t}$, a fraction.

233. SCH.—This proposition does not preclude the possibility of *surd* roots in this form of equation. These are possible.

---

234. Prop.—*An equation* f(x) = 0 (***229***) *of the* n*th degree, has* n *roots (if it has any* *), *and no more.*

DEM.—Let a be a root of $f(x) = 0$, which is of the nth degree. Dividing $f(x)$ by $x - a$ (***231***), we have $\varphi(x) = 0$, an equation of the $(n-1)$th degree.

Let b be a root of $\varphi(x) = 0$, and divide $\varphi(x)$ by $x - b$ (***231***). Call the quotient $\varphi'(x)$, whence $\varphi'(x) = 0$, an equation of the $(n-2)$th degree. In this way the degree of the equation can be diminished by division until, after $n-1$ divisions, there results $\varphi^n(x)$ † of the first degree, and the equation is $x - l = 0$. Therefore,

$$f(x) = (x - a)\,\varphi(x) = (x-a)(x - b)\,\varphi'(x) = (x - a)(x - b)(x - c)\,\varphi''(x)$$
$$= (x - a)(x - b)(x - c) \cdots\cdots (x - l) = 0;$$

i. e., $f(x)$ is resolvable into n factors, of the form $x - m$.

---

* We shall *assume* that every equation has a root real or imaginary; *i. e.*, that there is some form of expression which substituted for the unknown quantity will satisfy the equation. It is shown in works treating more largely upon the theory of equations, that the general form of a root is $\alpha + \beta\sqrt{-1}$. When $\beta = 0$, the root is real. The general demonstration of this proposition is too abstruse for an elementary treatise. That every equation of the form $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \cdots L = 0$ (***229***) has a real root when n is an odd number, and also when n is an even number if L be negative, is very simple. Thus, if n is odd, and L +, when x is made $-\infty$ the value of the first member is $-$; and when x is 0, the value is +. Hence while x passes from $-\infty$ to 0, the function changes sign, and hence must pass through 0; *i. e.*, for some value of x between $-\infty$ and 0, the equation is satisfied. In like manner, if L is $-$, when $x = 0$, the function is $-$, and when $x = +\infty$ the function is +. Hence some value of x between -0 and $+\infty$, satisfies the equation. It follows from this that in an equation of an odd degree, if the absolute term is +, there is at least one real, *negative* root; and if the absolute term is $-$, there is at least one real, *positive* root.

If n is even and L $-$, $x = 0$ makes the function $-$, and $x = \pm\infty$ makes it +. Hence while x passes from $-\infty$ to 0, the function changes sign from + to $-$, and there is at least one real, *negative* root; also, while x passes from 0 to $+\infty$, the function changes sign from $-$ to +, and there is at least one real, *positive* root. Therefore every equation of an even degree in which the absolute term is $-$, has at least two real roots, one negative, and one positive.

The difficulty occurs in proving that an equation of an even degree has a root when L is +. The roots of such an equation may be all imaginary,

† This is read "the nth φ function of x."

Now, as $x = a$, or $x = b$, or $x =$ any one of the quantities $a, b, c \cdots l$, will render $f(x)$ equal to 0, each one of these will satisfy the equation $f(x)=0$. Therefore this equation has n roots.

Again, since it is evident that we have resolved $f(x)$ into its *prime* factors with respect to x, there can be no other factor of the form $x-m$ in $f(x)$, hence no other root of $f(x)=0$, and this whether m is equal to one or more of the roots $a, b, c \cdots n$, or not. Therefore $f(x) = 0$ has only n roots.

235. COR. 1.—*The polynomial* $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \cdots L$, *or* $f(x), = (x-a)(x-b)(x-c) \cdots (x-l)$, *in which* $a, b, c \cdots l$ *are the roots of* $f(x) = 0$.

236. COR. 2.—*The equation* $f(x) = 0$ *may have* 2, 3, *or even* n *equal roots, as there is no inconsistency in supposing* $a = b$, $a = b = c$, *or* $a = b = c = \cdots l$, *in the above demonstration.*

237. COR. 3.—*Imaginary roots enter into equations having only real coefficients, in conjugate pairs* (**225**, PART I.)*; that is, if* $f(x)=0$ *has only real coefficients, if it has one root of the form* $\alpha + \beta\sqrt{-1}$, *it has another of the form* $\alpha - \beta\sqrt{-1}$; *or, if it has one of the form* $\beta\sqrt{-1}$, *it has another of the form* $-\beta\sqrt{-1}$.

This is evident, since only thus can $f(x)=(x-a)(x-b)(x-c) \cdots (x-n)$; that is, if one root, a for example, is $\alpha-\beta\sqrt{-1}$, there must be another of the form $\alpha+\beta\sqrt{-1}$, in order that the product of these two factors shall not involve an imaginary. Thus, $[x-(\alpha+\beta\sqrt{-1})] \times [x-(\alpha-\beta\sqrt{-1})]=x^2-2\alpha x+(\alpha^2+\beta^2)$, a real quantity. So also $(x-\beta\sqrt{-1})(x+\beta\sqrt{-1}) = x^2+\beta^2$, a real quantity. But if the assumed imaginary roots be not in conjugate pairs, the product of the factors $(x-a)(x-b)(x-c) \cdots (x-l)$ will involve imaginaries.

238. COR. 4.—*Hence an equation of an odd degree must have at least one real root; but an equation of an even degree does not necessarily have any real root.*

239. COR. 5.—*If an equation has a pair of imaginary roots, the known quantities entering into the equation may be so varied that the two imaginary roots shall first give place to two equal roots, and then these to two real and unequal roots*

As shown above, imaginary roots arise from real quadratic factors in $f(x)$. Let $x^2 - 2ax + b$ be such a quadratic factor, whence $x^2 - 2ax + b = 0$ satisfies $f(x) = 0$, and $a \pm \sqrt{a^2 - b}$ are the corresponding roots of $f(x) = 0$. Now, if $b > a^2$, these roots are imaginary. If, however, b diminishes or a increases (or both change thus together), when $b = a^2$ the two imaginary roots disappear and we have in their place two real roots, each a. If the same change in a and b

continues, so that a^2 becomes greater than b, the two real, equal roots in turn give place to two real, unequal roots. Now as a and b are functions of the known quantities of the equation $f(x) = 0$, such changes are evidently possible.

240. SCH. 1.—That an equation has a number of roots equal to its degree, is illustrated geometrically by the fact, that, if we write $y = f(x)$ and construct the locus, we shall always find that a straight line can be drawn so as to cut the locus in 1 point and only 1, if $f(x)$ is of the 1st degree (Ex's. 1–13, CHAP. II.); in 2 and only 2 points, if $f(x)$ is of the 2d degree (Ex's. 15–18, CHAP. II.); in 3 and only 3 points, if $f(x)$ is of the 3d degree (Ex's. 19–23, CHAP. II.); in 4 and only 4 points, if $f(x)$ is of the 4th degree (Ex's. 24–28, CHAP. II.), and specially illustrated by the line $X_1'X_2$, (*Fig.* 20), etc.

241. SCH. 2.—The fact that imaginary roots enter real equations in pairs is also beautifully illustrated by the loci of equations. Thus the equation $x^2-3x+5=0$ has two imaginary roots, and no real roots. Now, by reference to *Fig.* 9 of the preceding chapter, we see that the locus of $y=x^2-3x+5$ does not cut the axis of abscissas at all; *i. e.*, that no *real* value of x will give $f(x)=0$. But, if the equation were so modified as to make each ordinate only $\frac{11}{4}$ less than it now is, *i. e.*, if $y=x^2-3x+\frac{9}{4}$, we should have the same locus, but changed in position so as just to touch the axis of x, as in c, *thus giving* $f(x)=0$ *two real and equal roots.* If, again, we wrote $y=x^2-3x-3$, we should have the locus referred to the axis $A''X''$, and $f(x)=0$ would have two real and unequal roots. Thus we see, conversely, how two real, unequal roots can pass into two real and equal roots by a proper change in the equation, and how by a further change *two equal real roots disappear at a time,* passing into two imaginary roots as the equation changes form. All that is necessary in this change in the form of the equation is a proper change in the absolute term.

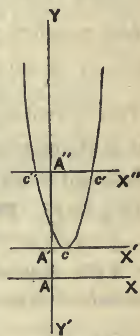

FIG. 22.

Again, consider *Fig.* 14, and the corresponding equation $y=x^3-6x^2+11x-6$. First we observe that as this locus cuts the axis of x three times, there are three real roots. Now change the absolute term -6 by allowing it to increase gradually, becoming $-5\frac{3}{4}$, $-5\frac{1}{2}$, -5, etc. We shall find that the axis of x moves down, and the two roots $A\,d$ and $A\,f$ approach equality, first becoming equal when the axis just touches the lowest point e of the curve, and then *both becoming imaginary together.*

Or, in conclusion, this matter is illustrated by the fact that whatever the degree of the equation $f(x)=0$, if we construct the locus of $y=f(x)$, we shall find that we can draw a straight line which will cut the curve in a number of points equal to the degree of the equation, and that if the line gradually moves from this position so as to cut the curve in any less number of points, it will always be found first to run two intersections together, corresponding to a change of two unequal roots into two equal roots, and then drop out *both* these intersections, corresponding to the introduction of two imaginary roots at a time.

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242. Prop.—*If the equation* f(x)=0 *has equal roots, the highest common divisor of* f(x) *and its differential coefficient,** f′(x), *being put equal to* 0, *constitutes an equation which has for its roots these equal roots, and no other roots.*†

DEM.—Let a be one of the m equal roots of $f(x)=0$, and let the other roots be b, c - - - - l; then $f(x)=(x-a)^m(x-b)(x-c)\,----\,(x-l)$ (*235*). Differentiating (*152*) and dividing by dx, we have

$$f'(x)=m(x-a)^{m-1}(x-b)(x-c)\,----\,(x-l)+(x-a)^m(x-c)\,----\,(x-l)+--\\----+(x-a)^m(x-b)(x-c)\,----+\text{etc.}$$

Now $(x-a)^{m-1}$ is evidently the highest common divisor of $f(x)$ and $f'(x)$, and $(x-a)^{m-1}=0$ is an equation having a for its root, and having no other.

In a similar manner, if $f(x)=0$ has two sets of equal roots, so that

$$f(x)=(x-a)^m(x-b)^r(x-c)(x-d)\,----\,(x-l),$$

differentiating and dividing by dx, we have

$$f'(x)=m(x-a)^{m-1}(x-b)^r(x-c)(x-d)\,----\,(x-l)+(x-a)^m r(x-b)^{r-1}(x-c)(x-d)\\----\,(x-l)\\+(x-a)^m(x-b)^r(x-d)\,----\,(x-n)+(x-a)^m(x-b)^r(x-c)\,----\,(x-l)+--\\----+(x-a)^m(x-b)^r(x-c)(x-d)\,----+\text{etc.}$$

Now the highest common divisor of $f(x)$ and $f'(x)$ is evidently $(x-a)^{m-1}(x-b)^{r-1}$. Putting this equal to 0, we have $(x-a)^{m-1}(x-b)^{r-1}=0$, an equation which is satisfied by $x=a$ and $x=b$, and by no other values.

Thus we may proceed in the case of any number of sets of equal roots.

243. SCH.—In searching for the equal roots of equations of high degree, it may be convenient to apply the process of the proposition several times. Thus, suppose that $f(x)=0$ has m roots each equal to a, and r roots each equal to b. Then the highest common divisor of $f(x)$ and $f'(x)$ is of the form $(x-a)^{m-1}(x-b)^{r-1}$; whence $(x-a)^{m-1}(x-b)^{r-1}=0$ is an equation having the equal roots sought. Therefore we can find the highest common divisor of $(x-a)^{m-1}(x-b)^{r-1}$, and its differential coefficient which will be of the form $(x-a)^{m-2}(x-b)^{r-2}$, and write $(x-a)^{m-2}(x-b)^{r-2}=0$, as an equation containing the roots sought. This process continued will cause one of the factors $(x-a)$ or $(x-b)$ to disappear and leave $(x-a)^{m-r}=0$, when $m>r$; $(x-b)^{r-m}=0$, when $r>m$; or $(x-a)(x-b)=0$, when $m=r$. From any one of these forms we can readily determine a root.

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* The differential coefficient of a function is sometimes called its first derived polynomial.

† The student must not suppose that the roots of $f(x)=0$, and its first differential coefficient $f'(x)=0$, are necessarily alike. $f'(x)=$ a series of *terms* some of which may be + and some −, and which may destroy each other, so as to render $f'(x)=0$, for other values of x than such as render $f(x)=0$, and not necessarily for any which do render $f(x)=0$, except the equal roots of the latter.

244. Prop.—*In an equation* f(x)=0, f(x) *will change sign when* x *passes through any real root, if there is but one such root, or if there is an odd number of such roots; but if there is an* EVEN *number of such roots,* f(x) *will not change sign.*

Let a, b, c - - - - e be the roots of $f(x)=0$, so that $f(x)=(x-a)(x-b)(x-c)$ - - - - $(x-e)=0$ (**235**). Conceive x to start with some value less than the least root, and continuously increase till it becomes greater than the greatest root. As long as x is less than the least root, all the factors $x-a, x-b$, etc., are negative; but when x passes the value of the least root, the sign of the factor containing that root will become +,* and if there is no other equal root, this factor will be the only one which will change sign. Hence the product of the factors will change sign. But if there is an even number of roots, each equal to this, an even number of factors will change sign; whence there will be no change in the sign of the function. If, however, there is an odd number of equal roots, the passage of x through the value of this root will cause a change of sign in an odd number of factors, and hence will change the sign of the function.

Finally, as it is evident that the signs of the factors, and hence of the function, will remain the same while x passes from one root to another, and in all cases changes or does not change as above when x passes through a root, the proposition is established.

245. SCH.—This proposition is illustrated by putting $y=f(x)$ and constructing the locus, as in the preceding chapter. Thus, Ex. 15, *Fig.* 6. In this case $y=f(x)=x^2+x-6$. The least root is -3. When x is less than -3, as -4, or $-3\frac{1}{2}$ (*anything* less than -3), y, or $f(x)$, is +. When x is -3, y, or $f(x)=0$, and the equation $f(x)=0$ is satisfied, and -3 is a root of the equation. When x becomes greater than -3, as -2, y, or $f(x)$, becomes negative, changing sign when x passes through the value of the root -3. As x increases, y, or $f(x)$, remains $-$, till x reaches $+2$, at which value of x, $y=f(x)=0$, and the equation $f(x)=0$ is satisfied. When x passes this value, becoming anything greater than 2, y, or $f(x)$, becomes +, *i. e.*, changes sign as x passes through the root 2. The same thing is illustrated by the loci in *Figs.* 7, 11, 12, 14, 15, and 18, with their corresponding equations.

That $f(x)$ does not change sign upon x's passing through the value of one of two equal roots of $f(x)=0$, is illustrated in *Fig.* 8 and its corresponding equation, Ex. 17. In this case $y=f(x)=x^2-4x+4$, and the equation $x^2-4x+4=0$ has two roots each equal to 2. Now when x is anything less than 2, y, *i. e.* $f(x)$, is +; when $x=2$, y, or $f(x)$, is 0, and the equation $f(x)=0$ is satisfied. But when x passes the value 2, $f(x)$ *does not* change sign; it remains +. The same truth is illustrated by the loci in *Figs.* 10, 13, 16, and 17, and their corresponding equations. *Fig.* 16 illustrates the case in which there are two pairs of equal roots.

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* Suppose c be the least root, and that c' is the next state of x greater than c; then $c'-c$ is +.

Ex. 29 will be found very instructive. The locus in *Fig.* 20 illustrates the case of 3 equal roots. Here $y = f(x) = x^5 + 4x^4 - 14x^2 - 17x - 6$. The least root is -3. When $x < -3$, $f(x)$ is $-$; when $x = -3$, $f(x) = 0$; when x passes -3, increasing, $f(x)$ changes from $-$ to $+$, and remains $+$ till $x = -1$, when it becomes 0, and changes sign as x passes -1, *notwithstanding there are equal roots.* But there is an *odd* number of such roots, viz., three.

Thus, if $X_1'X_1$ were to revolve about c' until it took the position $X'X$, the intersections b' and d' would run into c', the *three* intersections becoming *one.*

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246. Prop.—*Changing the signs of the terms of an equation containing the odd powers of the unknown quantity changes the signs of the roots.*

Dem.—If $x = a$ satisfies the equation $x^6 - Ax^4 + Bx^3 - Cx + D = 0$, we have $a^6 - Aa^4 + Ba^3 - Ca + D = 0$. Now changing the signs of the terms containing the odd powers of x, we have $x^6 - Ax^4 - Bx^3 + Cx + D = 0$. This is satisfied by $x = -a$, if the former is by $x = a$. For, substituting $-a$ for x, we have $a^6 - Aa^4 + Ba^3 - Ca + D = 0$, the same as in the first instance.

247. Cor.—*Changing the signs of the terms containing the even powers will answer equally well, since it amounts to the same thing; and if we are careful to put the equation in the complete form, changing the signs of the alternate terms will accomplish the purpose.*

Ill.—The *negative* roots of $x^3 - 7x + 6 = 0$, are the positive roots of $-x^3 + 7x + 6 = 0$, or of $x^3 - 7x - 6 = 0$ (0 being considered an *even* exponent); or, writing the equation $x^3 \pm 0x^2 - 7x + 6 = 0$, changing the signs of alternate terms, and then dropping the term with its coefficient 0, we obtain the same result.

Again, the negative roots of $x^6 - 7x^5 - 5x^4 + 8x^3 - 132x^2 + 508x - 240 = 0$, are the positive roots of $x^6 + 7x^5 - 5x^4 - 8x^3 - 132x^2 - 508x - 240 = 0$, or of $-x^6 - 7x^5 + 5x^4 + 8x^3 + 132x^2 + 508x + 240 = 0$.

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248. Prob.—*To evaluate* f(x) for any particular value of x, as x = a, more expeditiously than by direct substitution.*

Solution.—As $f(x)$ is of the form $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3}$ - - - - L, let it be required to evaluate $x^4 + Ax^3 + Bx^2 + Cx + D$ for $x = a$. Write the detached coefficients as below, with a at the right in the form of a divisor: thus

1	$+A$	$+B$	$+C$	$+D$	$\lvert a$
	a	$a^2 + Aa$	$a^3 + Aa^2 + Ba$	$a^4 + Aa^3 + Ba^2 + Ca$	
	$a + A$	$a^2 + Aa + B$	$a^3 + Aa^2 + Ba + C$	$a^4 + Aa^3 + Ba^2 + Ca + D$	

* This means to find the value of. Thus, suppose we want to find the value of $x^6 - 5x^5 + 2x^4 - 3x^3 + 6x^2 - x - 12$, for $x = 5$. We might substitute 5 for x, of course, and accomplish the end. But there is a more expeditious way, as the solution of this problem will show.

Having written the detached coefficients, and the quantity a for which $f(x)$ is to be evaluated as directed, multiply the first coefficient 1 by a, write the result under the second, and add, giving $a + A$. Multiply this sum by a, write the product under the third coefficient B, and add, giving $a^2 + Aa + B$. In like manner continue till all the coefficients (including the absolute term, which is the coefficient of x^0) have been used, and we obtain $a^4 + Aa^3 + Ba^2 + Ca + D$, which is the value of $f(x)$ for $x = a$.

ILLUSTRATION.—To evaluate $x^6 - 5x^5 + 2x^4 - 3x^3 + 6x^2 - x - 12$, for $x = 5$:

1	−5	+2	−3	+6	−1	−12	5
	5	0	10	35	205	1020	
	0	2	7	41	204	1008	

Now 1008 is the value of $x^6 - 5x^5 + 2x^4 - 3x^3 + 6x^2 - x - 12$, for $x = 5$; and it is easy to see that much labor is saved by this process.

We are now prepared for the solution of the following important practical problem:

***249. Prob.**—To find the commensurable roots of numerical higher equations.*

The solution of this problem we will illustrate by practical examples.

EXAMPLES.

1. Find the commensurable roots of $x^5 - 2x^4 - 15x^3 + 8x^2 + 68x + 48 = 0$, if it has any.

SOLUTION.—By (***232***), if this equation has any commensurable roots they are integral:—it can have no fractional roots.

Again, by (***230***), the roots of this equation with their signs changed are factors of 48. Now, the integral factors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24, 48. Hence, if the equation has commensurable roots, they are some of these numbers, with either the + or − sign. We will, therefore, proceed to evaluate $f(x)$ (*i. e.*, in this case $x^5 - 2x^4 - 15x^3 + 8x^2 + 68x + 48$), for $x = +1, x = -1, x = +2, x = -2$, etc., by (***248***), as follows:

1	−2	−15	+ 8	+68	+48	+1
	1	− 1	−16	− 8	60	
	−1	−16	− 8	60	108	

Hence we see that for $x = +1$, $f(x) = 108$, and $+1$ is not a root of $f(x) = 0$. Trying $x = -1$, we have

1	−2	−15	+ 8	+68	+48	−1
	−1	3	12	−20	−48	
	−3	−12	20	48	0	

Thus we see that for $x = -1, f(x) = 0$, and hence that -1 is a root of our equation.

We might now divide $f(x)$ by $x+1$ (***231***) and reduce the degree of the equation by unity. But it will be more expeditious to proceed with our trial. Let us therefore evaluate $f(x)$ for $x=+2$. Thus:

$$\begin{array}{rrrrrr|l} 1 & -2 & -15 & +8 & +68 & +48 & \underline{+2} \\ & 2 & 0 & -30 & -44 & +48 & \\ \hline & 0 & -15 & -22 & 24 & 96 & \end{array}$$

Hence for $x=+2$, $f(x)=96$, and $+2$ is not a root. Trying $x=-2$, we have

$$\begin{array}{rrrrrr|l} 1 & -2 & -15 & +8 & +68 & +48 & \underline{-2} \\ & -2 & 8 & 14 & -44 & -48 & \\ \hline & -4 & -7 & 22 & 24 & 0 & \end{array}$$

Hence for $x=-2$, $f(x)=0$, and -2 is a root. Trying $x=+3$, we have

$$\begin{array}{rrrrrr|l} 1 & -2 & -15 & +8 & +68 & +48 & \underline{+3} \\ & 3 & 3 & -36 & -84 & -48 & \\ \hline & 1 & -12 & -28 & -16 & 0 & \end{array}$$

Hence for $x=+3$, $f(x)=0$, and $+3$ is a root. Trying $x=-3$, we have

$$\begin{array}{rrrrrr|l} 1 & -2 & -15 & +8 & +68 & +48 & \underline{-3} \\ & -3 & 15 & 0 & -24 & -132 & \\ \hline & -5 & 0 & 8 & 44 & -84 & \end{array}$$

Hence for $x=-3$, $f(x)=-84$, and -3 is not a root. Trying $x=4$, we have

$$\begin{array}{rrrrrr|l} 1 & -2 & -15 & +8 & +68 & +48 & \underline{4^*} \\ & 4 & 8 & -28 & -80 & -48 & \\ \hline & 2 & -7 & -20 & -12 & 0 & \end{array}$$

Hence for $x=4$, $f(x)=0$, and 4 is a root.

We have now found four of the roots, viz., -1, -2, 3, and 4. Their product with their signs changed is 24. Hence, by (***230***) $48 \div 24 = 2$ is the other root with its sign changed, *i. e.* there are *two* roots -2.

That our equation had equal roots could have been ascertained by the principle in (***242***); but as the process of finding the H. C. D. is tedious, it is generally best to avoid it in practice.

2 to 12. Find the roots of the following:

(2.) $x^4 - x^3 - 39x^2 + 24x + 180 = 0$;

(3.) $x^3 + 5x^2 - 9x - 45 = 0$;

(4.) $x^3 + 2x^2 - 23x - 60 = 0$;

(5.) $x^4 - 3x^3 - 14x^2 + 48x - 32 = 0$;

(6.) $x^3 - 8x^2 + 13x - 6 = 0$;

* Of course it is not necessary to retain the + sign, as we have done in the preceding operations: it has been done simply for emphasis.

(7.) $x^4 - 11x^2 + 18x - 8 = 0$;*
(8.) $x^6 - 3x^5 + 6x^3 - 3x^2 - 3x + 2 = 0$;
(9.) $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0$;
(10.) $x^4 - 45x^2 - 40x + 84 = 0$;
(11.) $x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24 = 0$;
(12.) $x^6 - 7x^5 + 11x^4 - 7x^3 + 14x^2 - 28x + 40 = 0$.

13 to 20. Apply the process for finding equal roots (***242, 243***) to the following:

(13.) $x^3 + 8x^2 + 20x + 16 = 0$;
(14.) $x^3 - x^2 - 8x + 12 = 0$;
(15.) $x^3 - 5x^2 - 8x + 48 = 0$;
(16.) $x^4 - 11x^2 + 18x - 8 = 0$;
(17.) $x^4 + 13x^3 + 33x^2 + 31x + 10 = 0$;
(18.) $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0$;
(19.) $x^6 + 3x^5 - 6x^4 - 6x^3 + 9x^2 + 3x - 4 = 0$;
(20.) $x^7 + 5x^6 + 6x^5 - 6x^4 - 15x^3 - 3x^2 + 8x + 4 = 0$. (See ***243.***)

21 to 27. Having found all but two of the roots of each of the following by (***248***), reduce the equation to a quadratic by (***231***), and from this quadratic find the remaining roots:

(21.) $x^3 - 6x^2 + 10x - 8 = 0$;
(22.) $x^4 - 4x^3 - 8x + 32 = 0$;
(23.) $x^3 - 3x^2 + x + 2 = 0$;
(24.) $x^4 - 6x^3 + 24x - 16 = 0$;
(25.) $x^4 - 12x^3 + 50x^2 - 84x + 49 = 0$; †
(26.) $x^4 - 9x^3 + 17x^2 + 27x - 60 = 0$;
(27.) $x^5 - 4x^4 - 16x^3 + 112x^2 - 208x + 128 = 0$.

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28 to 34. Apply the processes of (***228***) to reduce the following to the form $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3}$ - - - - $L = 0$, before searching for roots:

(28.) $2x^3 - 3x^2 + 2x - 3 = 0$; ‡

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* In order to apply the process of evaluation, the coefficients of the missing powers must be supplied. Thus we have $1 + 0 - 11 + 18 - 8$.

† Apply the method for finding equal roots. The method of trial based upon (***230***) as applied by (***248***) is likely to lead to much unnecessary work when there are several equal roots, and all the others incommensurable.

‡ We have $x^3 - \frac{3}{2}x^2 + x - \frac{3}{2} = 0$. Put $x = \frac{y}{k}$, whence $\frac{y^3}{k^3} - \frac{3}{2k^2}y^2 + \frac{1}{k}y - \frac{3}{2} = 0$, or $y^3 - \frac{3k}{2}y^2 + k^2 y - \frac{3k^3}{2} = 0$. If now $k = 2$, we have $y^3 - 3y^2 + 4y - 12 = 0$, which can be solved as before, for one value of y, and the equation then reduced to a quadratic and solved for the other values. Finally, remembering that $x = \frac{1}{2}y$, we have the values of x required.

(29.) $3x^3 - 2x^2 - 6x + 4 = 0$;

(30.) $8x^3 - 26x^2 + 11x + 10 = 0$;

(31.) $x^4 - \frac{1}{2}x + \frac{3}{16} = 0$; (Look out for equal roots.)

(32.) $x^4 - 6x^3 + 9\frac{1}{2}x^2 - 3x + 4\frac{1}{2} = 0$;

(33.) $x = 19x^{-1} + \sqrt{1 - \frac{76}{x} - 403x^{-2}}$;

(34.) $\sqrt{x^2 - 3x + 22 - \frac{24}{x+1}} = 2(x+1)^{\frac{1}{2}}$.

250. By means of the property exhibited in (**235**) produce the equations whose roots are given in the following examples:

1. Roots 1, -3, 4.
2. Roots $\sqrt{2}$, $-\sqrt{2}$, -1, 3.
3. Roots 1, 2, 2, -3, 4.
4. Roots -3, $2+\sqrt{-1}$, $2-\sqrt{-1}$.
5. Roots 3, -2, -2, -2, 1.
6. Roots $\frac{2}{3}$, $\frac{1}{2}$, $-\frac{3}{5}$.
7. Roots $1 \pm \sqrt{-2}$, $2 \pm \sqrt{-3}$.
8. Roots $1\frac{1}{2}$, 2, $\sqrt{3}$, $-\sqrt{3}$.
9. Roots $\sqrt{-2}$, $-\sqrt{-2}$, $\sqrt{5}$, $-\sqrt{5}$.
10. Roots 10, -13, $\frac{1}{5}$, 1.
11. Roots $3-2\sqrt{3}$, $3+2\sqrt{3}$, $2-3\sqrt{-1}$, $2+3\sqrt{-1}$, 1, -1.

SECTION II.

SOLUTION OF NUMERICAL HIGHER EQUATIONS HAVING REAL, INCOMMENSURABLE, OR IRRATIONAL ROOTS.

251. As all equations having real roots have real coefficients* (**237**), and as all such can be reduced to the form $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} - - - - L = 0$, which we represent by $f(x) = 0$ (**229**), we shall consider this as the typical form. Moreover, since, if an equation of this character has equal roots, they can be determined by (**242, 243**), and the degree of the equation depressed by (**231**), we need only to consider the case in which $f(x) = 0$ has no equal roots.

* This is evident from the fact that $f(x)=(x-a)(x-b)(x-c) - - - - (x-n)=0$, in which if $a, b, c, - - - - n$ are real, no imaginary quantity will be found in the product of the binomials.

252. The best general method of approximating the real, incommensurable roots of such equations, is:

1st. To find the number and situation of such roots by STURM'S THEOREM and the method based on it.

2d. Having found the first figure or figures of such a root by Sturm's method, to carry forward the approximation to any required degree of accuracy by HORNER'S method of approximation.

These methods we will now proceed to develop.

STURM'S THEOREM AND METHOD.

253. Sturm's Theorem is a theorem by means of which we are enabled to find the *number* and *situation of the real roots* of any numerical equation with a single unknown quantity, real and rational coefficients, and without equal roots.*

ILL.—Thus, if we have the equation $x^3 - 7x + 7 = 0$, Sturm's Theorem enables us to determine that it has *three* real roots, *i. e.*, that all its roots are real. It also enables us to ascertain that one root lies between 1.3 and 1.4, another between 1.6 and 1.7, and the third between −3 and −4. Hence it shows us that the roots are 1.3+, 1.6+, and −3. with a decimal fraction.

254. SCH.—Of course it follows from the above that if the equation has commensurable (***227***) roots, Sturm's Theorem will enable us to find them, or even when the roots are not commensurable, it will enable us to find any number of initial figures. Thus in the equation $x^3 - 7x + 7 = 0$, we might by Sturm's Theorem find that the first root is 1.35689+; but it would be too tedious an operation to be of any practical utility, as will appear hereafter. We use this theorem only to find one or two of the initial figures, or, enough of the figures to enable us to distinguish between (separate) the roots. Thus, if we had an equation $f(x) = 0$, of which two roots were 2.356873+ and 2.3569564, we *might* use Sturm's Theorem to find the first five figures of each root, *i. e.*, to distinguish between (separate) the roots; but this is not the best practical method, as will appear hereafter.

255. The Sturmian Functions of $f(x) = 0$ (which has no equal roots) are functions obtained by treating $f(x)$ and its first differential coefficient $f'(x)$, as in the process of finding their H. C. D., except that in the process we must not multiply or divide by a negative quantity, and the signs of the several remainders must be

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* If the equation which we wish to solve has equal roots, they can be discovered by (***242, 243***), and the degree of the equation reduced by division.

changed before they are used as divisors. *These remainders with their signs changed are the Sturmian Functions.**

ILL.—Let the equation $f(x) = 0$ be $x^3 - 4x^2 - x + 4 = 0$. The first differential coefficient of $x^3 - 4x^2 - x + 4$ is $3x^2 - 8x - 1$. Dividing $x^3 - 4x^2 - x + 4$ by $3x^2 - 8x - 1$, first multiplying the former by 3 to avoid fractions,† exactly as in the process of finding the H. C. D., we find the first remainder of lower degree than our divisor to be $-19x + 16$. Hence $19x - 16$ is the first *Sturmian Function* of $x^3 - 4x^2 - x + 4$. Again, dividing $3x^2 - 8x + 1$ by $19x - 16$ (introducing such constant factors as necessary), we find the next remainder to be -2025. Hence 2025 is the second *Sturmian Function* of $x^3 - 4x^2 - x + 4$.

256. *Notation.*—As the function which constitutes the first member of our equation is represented by $f(x)$, and its first differential coefficient by $f'(x)$, we shall represent the *Sturmian Functions* by $f_1(x)$, $f_2(x)$, $f_3(x)$, etc., read "f sub 1 function of x," "f sub 2 function of x," etc., or simply "function sub 1," "function sub 2," etc.

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257. In any series of quantities distinguished as + and −, a succession of two like signs is called a *Permanence* of signs, and a succession of two unlike signs a *Variation*.

ILL.—In the function $x^6 - 3x^5 - 2x^4 + x^3 + x^2 + 5x - 4$, the signs of the terms are

$$+ \quad - \quad - \quad + \quad + \quad + \quad -.$$

The first and second constitute a variation; the second and third a permanence; the third and fourth a variation; the fourth and fifth a permanence; the fifth and sixth a permanence; and the sixth and seventh a variation. Thus, in this case, there are three permanences and three variations of signs.

So also if we have

$$\begin{aligned} f(x) &= x^5 - 7x^4 + 13x^3 + x^2 - 16x + 4, \\ f'(x) &= 5x^4 - 28x^3 + 39x^2 + 2x - 16, \\ f_1(x) &= 11x^3 - 48x^2 + 51x + 2, \\ f_2(x) &= 3x^2 - 8x + 4, \\ f_3(x) &= x - 2, \\ f_4(x) &= 0. \end{aligned}$$

For $x = 0$, $f(x) = +4$, or $f(x)$ is +; $f'(x)$ is −; $f_1(x)$ is +; $f_2(x)$ is +; $f_3(x)$ is −; and $f_4(x)$ being 0, its sign is not considered. Hence the series of signs of these functions, for $x = 0$, is + − + + −; and has three variations and one permanence.

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* I have thought it best not to include $f(x)$ and $f'(x)$ under the term Sturmian Functions. There seems to be no propriety in including them, inasmuch as they are not *peculiar* to Sturm's method; and by excluding them an important distinction is marked.

† We introduce or reject constant factors, just as in finding the H. C. D., only we may not introduce or reject *negative* factors, since the *signs* are an essential thing in these functions, and to multiply or divide by a negative number would change the signs of the functions.

For $x = 1$, we find $f(x)$, $-$; $f'(x)$, $+$; $f_1(x)$, $+$; $f_2(x)$, $-$; and $f_3(x)$, $-$; the series of signs being $- + + - -$. This gives two variations and two permanences.

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258. *Prop.*—*In the series of functions* $f(x)$, $f'(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$, $f_5(x)$ - - - - $f_n(x)$, *when* $f(x) = 0$ *has no equal roots, if* x *be conceived to pass through all possible real values, that is, to vary continuously, from* $-\infty$ *to* $+\infty$, *there will be no change in the number of variations and permanences in the signs of the functions, except when* x *passes through a root of* $f(x) = 0$; *and when it does pass through such a root, there will be a loss of one variation, and only one.**

DEM.—1st. Any change in x which does not cause some one of the functions to vanish, cannot cause any change in the signs of the functions; for no function can change its sign without passing through 0 or ∞, and from the form of the functions which we are considering, they cannot be ∞ for any finite value of x. (These functions are all of the form $Ax^n + Bx^{n-1} + Cx^{n-2}$ - - - - L.)

2d. *No two consecutive functions can vanish*, i. e., *become* 0, *for the same value of* x. For, in the process of producing the Sturmian functions from $f(x)$ and $f'(x)$, let the several quotients be represented by $q, q', q'', q''', q^{\text{iv}}$, etc.; whence, by the principles of division, we have

$$f(x) = f'(x)q \quad - f_1(x), \qquad (1)$$
$$f'(x) = f_1(x)q' \quad - f_2(x), \qquad (2)$$
$$f_1(x) = f_2(x)q'' \quad - f_3(x), \qquad (3)$$
$$f_2(x) = f_3(x)q''' - f_4(x), \qquad (4)$$
$$f_3(x) = f_4(x)q^{\text{iv}} - f_5(x), \qquad (5)$$
$$\text{etc.,} \qquad \text{etc.,} \qquad \text{etc.}$$

Now, if possible, suppose that some value of x, as $x = a$, renders two consecutive functions, as $f_2(x)$ and $f_3(x)$ each 0; that is, that they vanish simultaneously. Then, since from (4) we have $f_4(x) = f_3(x)q''' - f_2(x)$, $f_4(x) = 0$. So, also, from (5), $f_5(x) = f_4(x)q^{\text{iv}} - f_3(x)$, and $f_5(x) = 0$. Thus, as a consequence of the simultaneous vanishing of any two consecutive functions, we could show that all the functions would vanish. But as, by hypothesis, $f(x)$ and $f'(x)$ have no common divisor containing x, the last remainder found by the process of finding the H. C. D. cannot contain x, and hence cannot vanish for any value of x. It is therefore impossible that any two consecutive functions of the series should vanish for the same value of x (*i. e.*, simultaneously).

3d. *When any one of the functions, except* $f(x)$, *vanishes for a particular value*

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* This is the substance, though not the exact form, of the celebrated theorem discovered by M. STURM in 1829, and for which he received the mathematical prize of the French Academy of Sciences in 1834. It is certainly one of the most elegant discoveries in algebraic analysis made in modern times. It is a masterpiece of logic, and a monument to the sagacity of its discoverer. The original memoir containing this theorem is found in the "Mémoires présentés par divers savants à l'Académie des Sciences," Tom. VI., 1835.

of x, *the adjacent functions have opposite signs for this value.* Thus, if $f_3(x)$ is 0 for $x = b$, we have, from (4), $f_2(x) = -f_4(x)$, *i. e.*, the adjacent functions, neither of which can vanish for this value (2d), have opposite signs.

4th. *When any value of* x, *as* x = c, *causes any function except* f(x) *to vanish, the number of variations and permanences of the signs of the functions is the same for the preceding and the succeeding values of* x, i. e., *for* x = c − h *and* x = c + h, h *being an infinitesimal.* Thus, let $x = c$ render $f_3(x) = 0$; then, since the adjacent functions have opposite signs for this value of x, we have either $+f_2(x)$, 0, $-f_4(x)$, or $-f_2(x)$, 0, $+f_4(x)$, *i. e.*, +, 0, −, or −, 0, + (3d). Again, as neither of these adjacent functions vanishes for $x = c$ (2d), neither of them can change sign as x passes through c (1st). But $f_3(x)$ may or may not change sign as x passes through c (***244***); hence its signs may be ‡, =, ±, or ∓, the upper sign representing the sign of $f_3(x)$ just before x reaches c, and the lower its sign just after it passes, *i. e.*, for $x = c - h$, and $x = c + h$, respectively. Hence all the changes in signs which can occur are represented thus: + ‡ −, + = −, + ± −, + ∓ −, − ‡ +, − = +, − ± +, and − ∓ +. These taken in any way give simply *one* permanence and *one* variation. *Hence there can be no change in the number of variations and permanences of the signs of the functions, consequent upon the vanishing of any* INTERMEDIATE *function.*

5th. We are now to examine what changes, if any, are produced in the number of variations and permanences by the vanishing of an extreme function. And in the first place we repeat that the last function cannot vanish for any value of x, as it does not contain x. We have then to examine only the case in which $f(x)$ vanishes, *i. e.*, when x passes through any root of $f(x) = 0$. For this purpose let us develop $f(x + h)$ by Taylor's Formula, considering h an infinitesimal. Thus,

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2} + f'''(x)\frac{h^3}{\underline{|3}} + \text{etc.}$$

Now let r be any root of $f(x) = 0$, and substitute in this development r for x; whence

$$f(r+h) = f(r) + f'(r)h + f''(r)\frac{h^2}{2} + f'''(r)\frac{h^3}{\underline{|3}} + \text{etc.}$$

As r is a root of $f(x) = 0$, $f(r) = 0$; and as h is an infinitesimal, the terms containing its higher powers may be dropped (***151***, and foot-note). Thus we have $f(r+h) = f'(r)h$. Hence, as h is +, we see that $f(r+h)$, that is the function just after x passes a root, has the same sign as $f'(r)$, *i. e.* $f'(x)$ when x is at a root. But as $f'(x)$ does not vanish when $x = r$ (2d), $f'(r-h)$, $f'(r)$, and $f'(r+h)$ have the same signs.* Again, since, by hypothesis, $f(x) = 0$ has no equal roots, *it* changes sign when x passes through a root (***244***), *i. e.*, $f(r-h)$ and $f(r+h)$ have different signs. Thus, as $f(x)$ and $f'(x)$ have like signs just after x has passed a root, and $f(x)$ changes sign in passing, while $f'(x)$ does not, these functions have unlike signs just before x reaches a root,† and what was a variation in signs becomes a permanence; that is, a variation is lost.

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* That is, the first differential coefficient of $f(x)$ does not change sign when x passes through a root of $f(x)=0$.

† From this we see that the roots of $f'(x)=0$ are intermediate between those of $f(x)=0$, since if a, b, and c are roots of $f(x)=0$, in the order of their magnitudes, just before x reaches a

Finally, as we have before shown that as x passes through all values from $-\infty$ to $+\infty$, there can be no change in any of the functions except $f(x)$ which will affect the number of variations and permanences in the signs of the functions, there is *only* one variation lost when x passes through any root of $f(x)=0$.

259. Cor. 1.—*To ascertain the number of real roots of the equation* $f(x)=0$, *we substitute in* $f(x)$, $f'(x)$, $f_1(x)$, $f_2(x)$ - - - - $f_n(x^0)$,* $-\infty$ *for* x, *and note the number of variations of signs. Then substitute* $+\infty$ *for* x, *and note the number of variations. The excess of the number of variations in the former case over that in the latter indicates the number of real roots of the equation.*

This is a direct consequence of the proposition, since as x increases from $-\infty$, there is no change in the number of variations of the signs of the functions except when x passes through a root; and every time that it does pass through a root one variation is lost, and only one. But in passing from $-\infty$ to $+\infty$, x passes through all real values. Hence the excess of the number of variations for $x=-\infty$ over the number for $x=+\infty$ is equal to the total number of real roots.

260. Cor. 2.—*To ascertain how many real roots of* $f(x)=0$ *lie between any two numbers as* a *and* b, *substitute the less of the two numbers in* $f(x)$, $f'(x)$, $f_1(x)$, $f_2(x)$, *etc., and note the number of variations of signs. Then substitute the greater and note the number of variations. The excess of the number of variations in the former case over that in the latter indicates the number of real roots between the numbers* a *and* b.

This appears from the proposition in the same manner as Cor. 1.

261. Sch.—Since the total number of roots of an equation corresponds to the degree of the equation (**234**), if we ascertain as above the number of *real* roots in any given equation, the number of *imaginary* roots is known by implication.

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262. Prob.—*To compute the numerical values of* $f(x)$, $f'(x)$, $f_1(x)$, $f_2(x)$, *etc.*, i.e., *of any function of* x *for any particular value of* x, *when the function is of the form* $Ax^n + Bx^{n-1} + Cx^{n-2} + Dx^{n-3}$ - - - - P.

Solution.—Of course this can be done by merely substituting the proposed value of x in the function. But there is a more elegant and expeditious way, which we proceed to exhibit.

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root a, $f(x)$ and $f'(x)$ have different signs, and just *after*, they have like signs. But just before x reaches b, $f(x)$ and $f'(x)$ have unlike signs, and as $f(x)$ cannot have changed sign, the sign of $f'(x)$ must have changed; *i.e.*, x must have passed through a root of $f'(x)=0$, in passing from a to b. In like manner it may be shown that a root of $f'(x)$ lies between each two consecutive roots of $f(x)=0$. This makes $f'(x)=0$ have one root less than $f(x)=0$, as it should.

* By this notation is meant the nth or last of the Sturmian functions, in which x does not appear; or, what is the same thing, that in which the exponent of x is 0.

Thus, let it be required to evaluate $Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$ for $x = a$. Multiply A by a and add the product to B. Multiply this sum by a and add the product to C. Multiply this sum by a and add the product to D. Continue this operation till all the coefficients have been involved and the absolute term added. The last sum is the value of the function when a is substituted for x, as will appear from considering the following:

$$\begin{array}{l}
A \\
a \\
\hline
Aa + B \\
a \\
\hline
Aa^2 + Ba + C \\
a \\
\hline
Aa^3 + Ba^2 + Ca + D \\
a \\
\hline
Aa^4 + Ba^3 + Ca^2 + Da + E \\
a \\
\hline
Aa^5 + Ba^4 + Ca^3 + Da^2 + Ea + F.
\end{array}$$

This is evidently the value of the function when a is substituted for x.

N. B.—1. If the function is not complete, *i. e.*, if it lacks any of the successive powers of x, care must be taken to supply the lacking coefficients with 0's. Thus the coefficients of $x^4 - 2x^2 + 5$ are to be considered as $1, 0, -2, 0$, and 5 (which may be called the coefficient of x^0).

2. When the numbers involved are small the operation can be performed mentally.

Ex. 1. Evaluate $257x^3 - 312x^2 + 1553x - 5247865$ for $x = 342$.

OPERATION.

$$\begin{array}{r}
257 \\
342 \\
\hline
514 \\
1028 \\
771 \\
\hline
87894 \\
-\quad 312 \\
\hline
87582 \\
342 \\
\hline
175164 \\
350328 \\
262746 \\
\hline
29953044 \\
1553 \\
\hline
29954597 \\
342 \\
\hline
59909194 \\
119818388 \\
89863791 \\
\hline
10244472174 \\
-\quad 5247865 \\
\hline
10239224309
\end{array}$$

The value required.

Ex. 2. Evaluate $x^4 - 3x^3 + 5x - 20$ for $x = 2$, performing the operation mentally.

Examples of the Use of Sturm's Method.

1. Find the number and situation of the real roots of $x^3 - 4x^2 - 6x + 8 = 0$.

Sug's.—If the student has attended carefully to what precedes, he will have no difficulty in determining that

$$f(x) = x^3 - 4x^2 - 6x + 8;$$
$$f'(x) = 3x^2 - 8x - 6;$$
$$f_1(x) = 17x - 12;$$
and
$$f_2(x^0) = 1467.$$

Now, for $x = -\infty$, we have $f(x)-$, $f'(x)+$, $f_1(x)-$, and $f_2(x^0)+$; *i. e.*, the signs of the functions are $- + - +$. There are therefore three variations.

Again, when $x = +\infty$, the signs are $+ + + +$, giving no variations. Hence the number of real roots is $3 - 0 = 3$; *i. e.*, they are all real.

To find the situation of these roots we observe that for $x = 0$, the signs of the functions are $+ - - +$, giving two variations, or one less than $-\infty$ gives. Hence there is one root between $-\infty$ and 0; *i. e.*, one negative root. The other two must of course be positive. We will first seek the situation of this negative root. Evaluate by (***262***).

For	$x = 0$,	the signs of	the functions	are			$+ - - +$.
"	$x = -1$,	"	"	"	"	"	$+ + - +$.
"	$x = -2$,	"	"	"	"	"	$- + - +$.*

Hence, as one variation is lost when x passes from -2 to -1, there is one root between -1 and -2; *i. e.*, the negative root is -1 and a fraction.

In like manner seeking the situation of the positive roots, evaluating the functions by (***262***), we have

For	$x = 0$,	the signs	$+ - - +$,	2	variations.
"	$x = 1$,	" "	$- - + +$,	1	"
"	$x = 2$,	" "	$- - + +$,	1	"
"	$x = 3$,	" "	$- - + +$,	1	"
"	$x = 4$,	" "	$- + + +$,	1	"
"	$x = 5$,	" "	$+ + + +$,	0	"

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* The evaluation of these functions is most elegantly and expeditiously effected by (***262***). Thus for $x = -2$ we have

1	−4	−6	+8	| −2
	−2	12	−12	
	−6	6	$-4 = f(x)$	

3	−8	−6	| −2
	−6	28	
	−14	$22 = f'(x)$	

When the value of x for which we are evaluating is small, and the coefficients also small, this process can be carried on mentally without writing, and should be so done.

Therefore, as one variation is lost when x passes from 0 to 1, there is one root between 0 and 1, *i.e.*, an incommensurable decimal. Again, one variation is lost when x passes from 4 to 5; hence the other root lies between 4 and 5, or is 4 and an incommensurable decimal.

263. Sch. 2.—It is usually unnecessary to find $f_n(x^0)$ (the last of the Sturmian functions), since its sign, which is all that is important, can be determined by inspection from the next to the last function and the preceding divisor. Thus, if we were to divide $3x^2+22x-102$ by $122x-393$, first multiplying the former by 122, it would be clear that the remainder would be $-$, without going through the operation. Hence $f_n(x^0)$ would be $+$.

2 to 7. Find the number and situation of the real roots of the following:

(2.) $x^3+6x^2+10x-1=0$; (5.) $x^5-2x^4+x^3-8x+6=0$;
(3.) $x^3-6x^2+8x+40=0$; (6.) $x^4-4x^3+x^2+6x+2=0$;
(4.) $x^4-4x^3-3x+23=0$; (7.) $x^4+2x^3+17x^2-20x+100=0$.

264. Sch. 3.—In case the equation has *equal* roots, we shall detect them in the process of producing the Sturmian functions, since in such a case the division will become exact at some stage of the process, and the last Sturmian function will be 0. Having thus discovered that the equation has equal roots, we might divide out the factors containing them, and then operate on the depressed equation as above for the unequal roots. But it is not necessary to depress the degree of the equation, since the several Sturmian functions will have the same variations of signs in either case for any particular value of x. This arises from the fact that the common divisor of $f(x)$ and $f'(x)$, which contains the equal roots, is a factor of each of the Sturmian functions, and hence its presence or absence will not affect their signs for any particular value of x if the common factor is $+$ for this value, and will change the signs of *all* if it is $-$; but in either case the variations of signs will not be affected.

8. Find the number and situation of the *unequal* real roots of $x^5-6x^4+7x^3+22x^2-60x+40=0$, without depressing the equation.

Sug's.—Forming the required functions, we have

$$f(x)=x^5-6x^4+7x^3+22x^2-60x+40;$$
$$f'(x)=5x^4-24x^3+21x^2+44x-60;$$
$$f_1(x)=37x^3-228x^2+468x-320;$$
$$f_2(x)=x^2-4x+4;$$
$$f_3(x)=0.$$

Now $f_2(x)$ is a factor of $f(x)$, $f'(x)$, and $f_1(x)$, and removing it from *all*, we shall have the following functions, which may be used instead of the Sturmian functions derived from the depressed equation:

$$f(x) = x^3 - 2x^2 - 5x + 10;$$
$$f'(x) = 5x^2 - 4x - 15;$$
$$f_1(x) = 37x - 80;$$
$$f_2(x) = 1.$$

Hence, since the *signs* of these two sets of functions evaluated for any particular value of x will be the same, either set may be used at pleasure.

Thus either set gives

For $x = -\infty$, $- + - +$;

and for $x = +\infty$, $+ + + +$.

Therefore there are two *unequal* real roots of $f(x) = 0$; and from the existence of the factor $(x-2)^2$ in $f(x)$ and $f'(x)$, we know that there are *three equal roots*, each 2.

The situation of the unequal roots can now be found as before.

9 to 12. Find the number and situation of the real roots of the following:

(9.) $x^5 - 7x^4 + 13x^3 + 11x^2 - 66x + 72 = 0$;
(10.) $x^5 - 18x^3 - 28x^2 + 24x + 48 = 0$;
(11.) $x^4 - 4x^3 + x^2 + 20x + 13 = 0$;
(12.) $x^5 - 10x^3 + 6x + 1 = 0$.

265. SCH. 4.—Elegant as the method of Sturm is, and perfectly as it accomplishes its object, the labor of producing the functions required and evaluating them, especially when the roots are large and widely separated, is so great as to deter us from its use when less laborious methods will serve the purpose. *In a great majority of practical cases in which there are no equal roots, the principle that* f(x) *changes sign when* x *passes through a root of* f(x) = 0 *will enable us to determine the situation of the roots with far less labor than Sturm's Theorem.* Often a simple inspection of the equation will determine the near value of a root. Methods are usually given for ascertaining the limits (as they are improperly called) of the roots of an equation, from the coefficients. But these are of little practical value.*

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* For example, the two following, which are most frequently given:

1. *In any equation the greatest negative coefficient with its sign changed and increased by unity is a* SUPERIOR LIMIT *of the roots.*

2. *In any equation unity added to that root of the greatest negative coefficient with its sign changed, whose index is equal to the difference of the exponents of the first term, and the first negative term is a* SUPERIOR LIMIT.

Now consider the equation $x^3 + x^2 - 500 = 0$. By the first rule the superior limit of a root is 501, and by the second $\sqrt{500} + 1$, or 23 +. Now the fact is, the greatest root is 7.6 +. Again, by 1, the superior limit of the roots of $x^3 - 3x^2 - 46x - 72 = 0$ is 73; and by 2 it is the same. But the greatest root is 9.

13. Find by inspection, and also by Sturm's method, the situation of the roots of the equation $x^3 + x^2 - 500 = 0$.

Sug's.—Let the student apply Sturm's method. The following is a solution by inspection:

Since $x = \sqrt[3]{500 - x^2}$, there is a + root less than $\sqrt[3]{500}$, or less than 8. Now, trying 7, we have

$$\begin{array}{rrrrl} 1 & +1 & 0 & -500 & \underline{|\,7} \\ & 7 & 56 & 392 & \\ \hline & 8 & 56 & -108, & \textit{i.e.,}\ f(x) \text{ is } -. \end{array}$$

Trying 8,

$$\begin{array}{rrrrl} 1 & +1 & 0 & -500 & \underline{|\,8} \\ & 8 & 72 & 576 & \\ \hline & 9 & 72 & 76, & \textit{i.e.,}\ f(x) \text{ is } +. \end{array}$$

There is therefore a root between 7 and 8.

Also from the relation $x = \sqrt[3]{500 - x^2}$, or from the operations above, we see that there is no other positive root; since $f(x)$ evaluated for any positive quantity less than 7 would certainly be $-$, and for anything greater than 8, $+$.

Finally, that there can be no negative root is evident, since $\sqrt[3]{500 - x^2}$ cannot be negative until $x^2 > 500$, but then $\sqrt[3]{500 - x^2} < \sqrt[3]{-x^2}$, and $\sqrt[3]{-x^2}$ is always $< x$. Hence for x negative we can never have $x = \sqrt[3]{500 - x^2}$. Therefore our equation has one real and two imaginary roots.

Note.—The advantage of this method of inspection over Sturm's method, in this case, will not be fully seen unless the student observes that all this can be done mentally, without writing a single figure.

14. Find by inspection, and also by Sturm's method, the number and situation of the real roots of $x^3 + x^2 + x - 100 = 0$.

Sug's.—A mere glance should show that there can be but one positive root, and that that is less than 5. In like manner writing $x^3 - x^2 + x + 100 = 0$, or $x^3 + x + 100 = x^2$, we see that no positive value of x can satisfy the equation; for when x is less than 1, of course the first member is greater than the second, and when x is greater than 1, x^3 itself is greater than x^2.

15. Find, by inspecting the changes of sign of $f(x)$ for varying values of x, the situation of the roots of $x^3 - 3x - 1 = 0$, and also by Sturm's method.

16. Find by inspection the situation of the roots of $x^3 - 22x - 24 = 0$.

Sug's.—Writing $x(x^2 - 22) = 24$, we see that any positive value of x which satisfies this must make $x^2 > 22$, that is, must be greater than 4. But 5 makes $x(x^2 - 22) = 15$, and 6 makes it 84. Moreover, it is evident that no number greater than 6 will satisfy the equation. Seeking for negative roots, we write $x^3 - 22x + 24 = 0$; and then $x(x^2 - 22) = -24$. To satisfy this, x^2 must be less than 22, or $x < 5$. For $x = 0$, $f(x)$ is +; for $x = 1$, $f(x)$ is +; for $x = 2$, $f(x)$ is −. Hence a root of the given equation between −1 and −2. Finally, for $x = 3$, $f(x)$ is −; but for $x = 4$, $f(x) = 0$. Hence a root of the given equation is −4.

17. Determine the situation of the roots of $x^5 - 10x^3 + 6x + 1 = 0$, by examining the changes of sign of $f(x)$.

SUG'S.—For $x = 0$, $f(x)$ is +; for $x = 1$, $f(x)$ is −; for $x = 2$, $f(x)$ is −; for $x = 3$, $f(x)$ is −; for $x = 4$, $f(x)$ is +; and will evidently remain +, as x advances beyond 4. This is seen from the following:

1	0	−10	0	+ 6	+1	4
	4	16	24	+96	408	
	4	6	24	102	409	

Now any positive number greater than 4 would destroy the −10 in this process, and give the sum at that point greater than 6, and hence the aggregate would rapidly increase. Thus notice, when 3 is substituted, we have

1	0	−10	0	+6	+1	3
	3	9	−3	−9	−9	
	3	− 1	−3	−3	−8	

Now 3 is not large enough to destroy the −10; but every number larger than 4 will destroy it.

To examine for negative roots we write $x^5 - 10x^3 + 6x - 1 = 0$. In this, for $x = 0$, $f(x)$ is −; for $x = 1$, $f(x)$ is −; for $x = 2$, $f(x)$ is −; for $x = 3$, $f(x)$ is −; but for $x = 4$, and all numbers greater than 4, $f(x)$ is +.

We have now found that there are certainly three roots between −4 and +4, and none beyond these limits either way. *But it is not safe to conclude that the other two roots are imaginary.* The fact is, *they are not.* How, then, are we to find them? Sturm's method is thought to possess particular advantage in saving us from such erroneous conclusions, and enabling us to find the situation of *all* the real roots with infallible certainty. And certainly it does do this; but let us see if we cannot do it, in this instance at least, as readily without that method. It will be observed that we know only that −3 is the initial figure of one root, and +3 of another. The initial digit of the root between 0 and +1 we have not found. Let us seek it. For $x = 0$, $f(x)$ is +; and by trying $x = .1$, $x = .2$, we should at once see that $f(x)$ changes very slowly, and as when $x = 1$, $f(x)$ is only −2, we should be led to try numbers *near* 1. Trying $x = .8$, we would find that $f(x)$ is +, and for $x = .9$, $f(x)$ is −. Hence .8 is the initial figure of the root lying between 0 and +1.

We now know the initial figures of three of the roots. But where are the other two roots? If they are *real* we know that they lie between −4 and +4, as we have seen above that no root can lie beyond these limits. Moreover, as the function changes value rapidly beyond 1, and slowly between −1 and 1, it would naturally be suggested that there may be *two* changes of sign between 0 and + 1, or 0 and − 1. Evaluating $f(x) = x^5 - 10x^3 + 6x + 1$ for .1, .2, .3, etc., we soon see that *it* will not change sign for values of x between 0 and + 1. Evaluating $f(x) = x^5 - 10x^3 + 6x - 1$ for .1, .2, .3, etc., we find that the other roots are between 0 and − 1, and that their initial digits are −.1 and −.6.

18 to 23. Find by inspection, by the change in sign of $f(x)$, or by Sturm's method, the number and situation of the real roots of the following:

(18.) $x^3 - 3x^2 - 4x + 11 = 0$;
(19.) $x^3 - 2x - 5 = 0$;
(20.) $x^4 - 4x^3 - 3x + 23 = 0$;
(21.) $x^3 + 11x^2 - 102x + 181 = 0$;
(22.) $x^3 - 17x^2 + 54x = 350$;
(23.) $x^3 + 2x^2 + 3x - 13089030 = 0$.*

266. Sch. 5.—If we have an equation in which, when cleared of fractions, the coefficient of the highest power of x is not unity, it may be transformed by (**228**) into one having such coefficient. *But this is not necessary in order to the application of Sturm's method, as it is not required by anything in the demonstration of that theorem that the coefficients should be integral.*

24 to 31. Find by Sturm's method the number and situation of the real roots of the following:

(24.) $2x^3 + 3x^2 - 4x - 10 = 0$;
(25.) $x^3 - 18\frac{1}{12}x + 29\frac{53}{108} = 0$; †
(26.) $8x^3 - 36x^2 + 46x - 15 = 0$;
(27.) $4x^3 - 12x^2 + 11x - 3 = 0$;
(28.) $3x^4 - 4x^2 + 2x - 1000 = 0$;
(29.) $7x^2 - 83x + 187 = 0$;
(30.) $x^3 - 1\frac{4}{5}x^2 - 1\frac{2}{5}x = 440$;
(31.) $x^3 - \frac{3}{5}x^2 - \frac{2}{5}x = 312$.

Horner's Method of Solution.‡

267. Horner's method of solving numerical equations is a method of finding the incommensurable roots of such equations to any re-

* Observe that neglecting the terms $2x^2 + 3x$, which, since x is large, are small as compared with x^3, we have $x^3 = 13089030$, or x lies between 200 and 300 *probably*.

† Clear of fractions first.

‡ Among the many methods discovered, and doubtless to be discovered, for this purpose, it is scarcely possible that Horner's should be superseded, since the solution of such an equation will certainly require the extraction of a root corresponding to the degree of the equation; and the labor required by Horner's method is not greater than that required to extract this root. Nor is this merely a method of approximation, except as any method for *incommensurable* roots is necessarily a method of approximation. If the root can be expressed exactly in the decimal notation, or by means of a repeating decimal, this process effects it. The method was first published by W. G. Horner, Esq., of Bath, England, in 1819, about fifteen years before Sturm's Theorem was published.

quired degree of approximate accuracy. It is based upon the two following problems and proposition:

268. Prob.—*To transform an equation, as* f(x) = 0, *into another whose roots shall be* a *less than those of the given equation.*

SOLUTION.—Let $x=a+x_1$, whence $x_1=x-a$, and we have $f(x)=f(a+x_1)=0$, or $0=f(a+x_1)$. Developing the latter by Taylor's Formula, we have $0=f(a+x_1)=f(a)+f'(a)x_1+f''(a)\frac{x_1^2}{2}+f'''(a)\frac{x_1^3}{\underline{|3}}+f^{\text{iv}}(a)\frac{x_1^4}{\underline{|4}}+$ etc., or

$0=f(a)+f'(a)x_1+f''(a)\frac{x_1^2}{2}+f'''(a)\frac{x_1^3}{\underline{|3}}+f^{\text{iv}}(a)\frac{x_1^4}{\underline{|4}}$, etc., as the required equation.

269. SCH.—The meaning of this may be stated thus: The absolute term of the transformed equation is the value of $f(x)$ when a is substituted for x; the coefficient of the first power of the unknown quantity, x_1, in the new equation is the first differential coefficient of $f(x)$, when a is substituted for x in this coefficient; the coefficient of the second power of x_1 is $\frac{1}{\underline{|2}}$ the second differential coefficient of $f(x)$, when a is substituted for x; etc.

EX.—From $5x^4-12x^3+3x^2+4x+5=0$ deduce a new equation whose roots shall be each less by 2 than the roots of this.

SOLUTION.

$$f(x)=5x^4-12x^3+3x^2+4x+5 \underset{x=a=2^*}{\quad} =9=f(a).$$

$$f'(x)=20x^3-36x^2+6x+4 \underset{x=2}{\quad} =32=f'(a).$$

$$f''(x)=60x^2-72x+6 \underset{x=2}{\quad} =102=f''a. \quad \therefore \tfrac{1}{2}f''(a)=51.$$

$$f'''(x)=120x-72 \underset{x=2}{\quad} =168=f'''(a). \quad \therefore \frac{1}{\underline{|3}}f'''(a)=28.$$

$$f^{\text{iv}}(x)=120 \underset{x=2}{\quad} =120=f^{\text{iv}}(a). \quad \therefore \frac{1}{\underline{|4}}f^{\text{iv}}(a)=5.$$

Hence $0=9+32x_1+51x_1^2+28x_1^3+5x_1^4$, or $5x_1^4+28x_1^3+51x_1^2+32x_1+9=0$, is an equation whose roots are 2 less than the roots of the given equation, since $x_1=x-2$.

270. Prob.—*To compute the numerical values of* f(a), f′(a), ½f″(a), $\frac{1}{\underline{|3}}$f‴(a), $\frac{1}{\underline{|4}}$f$^{\text{iv}}$(a), *etc., from* f(x), *when* f(x) *has the form* $Ax^n+Bx^{n-1}+Cx^{n-2}+Dx^{n-3}$ - - - - P.

SOLUTION.—Let $f(x)=Ax^4+Bx^3+Cx^2+Dx+E$; whence, forming $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{\text{iv}}(x)$, and substituting a for x, we have

* The meaning of this notation is that x is made equal to 2 in the function whence results the following value.

$$f(a) = Aa^4 + Ba^3 + Ca^2 + Da + E;$$
$$f'(a) = 4Aa^3 + 3Ba^2 + 2Ca + D;$$
$$\tfrac{1}{2}f''(a) = 6Aa^2 + 3Ba + C;$$
$$\tfrac{1}{|\underline{3}}f'''(a) = 4Aa + B;$$
$$\tfrac{1}{|\underline{4}}f^{\text{iv}}(a) = A.$$

Now, we may compute these as follows:

OPERATION.

$$\begin{array}{llll|l}
A \quad +B & +C & +D & +E & \underline{a} \\
\quad\ \underline{\ \ Aa} & \underline{\quad Aa^2+Ba} & \underline{\quad Aa^3+Ba^2+Ca} & \underline{\quad Aa^4+Ba^3+Ca^2+Da} & \\
Aa+B & Aa^2+Ba+C & Aa^3+Ba^2+Ca+D & Aa^4+Ba^3+Ca^2+Da+E=f(a). & \\
\underline{Aa} & \underline{2Aa^2+Ba} & \underline{3Aa^3+2Ba^2+Ca} & & \\
2Aa+B & 3Aa^2+2Ba+C & 4Aa^3+3Ba^2+2Ca+D=f'(a). & & \\
\underline{Aa} & \underline{3Aa^2+Ba} & & & \\
3Aa+B & 6Aa^2+3Ba+C=\tfrac{1}{2}f''(a). & & & \\
\underline{Aa} & & & & \\
4Aa+B=\tfrac{1}{|\underline{3}}f'''(a). & & & &
\end{array}$$

EXPLANATION.

Write the coefficients in a line with a at the right. Multiply A by a, write the product Aa under B, and add it to B, giving $Aa+B$. Multiply this result by a, write it under C, and add it to C, giving Aa^2+Ba+C. Proceed in this manner until all the coefficients have been involved. The last sum is $f(a)$.

Now repeat the process, by multiplying A by a, writing the product under $Aa+B$, and adding, thus producing $2Aa+B$. Multiply this sum by a, and add it to Aa^2+Ba+C. Continue the process till all the preceding sums *except the last* have been involved, thus producing $4Aa^3+3Ba^2+2Ca+D=f'(a)$. $\tfrac{1}{2}f''(a)$, $\tfrac{1}{|\underline{3}}f'''(a)$, (and others when necessary), are produced in a similar manner.

N. B.—It is to be observed that the coefficient of the highest power of x is the same in both equations.

EXAMPLES.

1. Transform $3x^4-4x^3+7x^2+8x-12=0$ into another equation each of whose roots shall be 3 less than the roots of this.

Solution.—Arranging the coefficients and proceeding as in the above solution, we ha e the following:

OPERATION.

3	-4	$+7$	$+8$	-12	$\lvert 3$
	9	15	66	222	
	5	22	74	$210 = f(3)$	
	9	42	192		
	14	64	$266 = f'(3)$		
	9	69			
	23	$133 = \frac{1}{2}f''(3)$			
	9				
	$32 = \frac{1}{\lfloor 3}f'''(3)$.				

Hence the transformed equation is

$$3x_1^4 + 32x_1^3 + 133x_1^2 + 266x_1 + 210 = 0.$$

2. Transform $3x^4 - 13x^3 + 7x^2 - 8x - 9 = 0$ into another equation whose roots shall be less by 3 than the roots of this.

The new equation is $3x^4 + 23x^3 + 52x^2 + 7x - 78 = 0$.*

3. Transform $x^5 + 2x^3 - 6x^2 - 10x + 8 = 0$ into another equation whose roots shall be 2 less than the roots of this.

PROCESS.

1	0	$+2$	-6	-10	$+8$	$\lvert 2$
	2	4	12	12	4	
	2	6	6	2	12	
	2	8	28	68		
	4	14	34	70		
	2	12	52			
	6	26	86			
	2	16				
	8	42				
	2					
	10					

∴ The equation is $x^5 + 10x^4 + 42x^3 + 86x^2 + 70x + 12 = 0$.

4. Transform $x^5 - 6x^4 + 7.4x^3 + 7.92x^2 - 17.872x - .79232 = 0$ into another equation whose roots shall be each less by 1.2 than the roots of this.

5. Transform $x^3 - 2x^2 + 3x + 4 = 0$ into another equation whose roots shall be 1.7 less than the roots of this.

6. Transform $x^3 + 11x^2 - 102x + 181 = 0$ into another equation whose roots shall be 3 less than the roots of this equation: transform the

* For convenience in reading and writing, it is customary to omit the subscripts which distinguish the unknown quantity in the transformed equation from that in the given equation. But it should be borne in mind that the unknown quantities *are* different.

resulting equation into another whose roots shall be .2 less than the roots of the last: transform this equation into another whose roots shall be .01 less than those of the last: transform this into another whose roots shall be .003 less than its roots.

OPERATION.

1	+11	−102	+181	3
	3	42	−180	
	14	−60	1*	.2
	3	51	−.992	
	17	−9*	.008†	.01
	3	4.04	−.006739	
	20*	−4.96	.001261‡	.003
	.2	4.08	−.001217403	
	20.2	−.88†	.000043597§	
	.2	.2061		
	20.4	−.6739		
	.2	.2062		
	20.6†	−.4677‡		
	.01	.061899		
	20.61	−.405801		
	.01	.061908		
	20.62	−.343893§		
	.01			
	20.63‡			
	.003			
	20.633			
	.003			
	20.636			
	.003			
	20.639§			

EXPLANATION.

* These, together with the first, are the coefficients of the equation whose roots are 3 less than those of the given equation. The equation written out is $x^3+20x^2-9x+1=0$. (A). But, instead of rewriting these coefficients for the second transformation, we operate upon them just as they stand.

† These, together with the first, are the coefficients of the equation whose roots are .2 less than those of (A), and consequently 3.2 less than those of the given equation. This equation written out is $x^3+20.6x^2-.88x+.008=0$. (B). But instead of rewriting these coefficients we effect the next transformation upon them just as they stand.

‡ These, together with the first (which remains the same in all), are the coefficients of the equation whose roots are .01 less than the roots of (B), .21 less than the roots of (A), and 3.21 less than the roots of the given equation. This equation is $x^3+20.63x^2-.4677x+.001261=0$. ($C$).

§ These are the coefficients of the equation whose roots are .003 less than those of (C), .013 less than those of (B), .213 less than those of (A), and 3.213 less than those of the given equation. The last transformed equation is $x^3+20.639x^2-.343893x+.000043597=0$.

7. Transform, as above, the equation $x^4-12x^2+12x-3=0$, successively, into equations whose roots shall be 2 less, 2.8 less, and 2.85 less than the roots of the given equation.

OPERATION.

1	0	−12	+12	−3	2.85
	2	4	−16	−8	
	2	−8	−4	−11*	
	2	8	0	8.9856	
	4	0	−4*	−2.0144†	
	2	12	15.232	1.71940625	
	6	12*	11.232	−.29499375‡	
	2	7.04	21.376		
	8*	19.04	32.608†		
	.8	7.68	1.780125		
	8.8	26.72	34.388125		
	.8	8.32	1.808375		
	9.6	35.04†	36.196500‡		
	.8	.5625			
	10.4	35.6025			
	.8	.5650			
	11.2†	36.1675			
	.05	.5675			
	11.25	36.7350‡			
	.05				
	11.30				
	.05				
	11.35				
	.05				
	11.40‡				

Hence the successive equations are,

The Primitive, $x^4-12x^2+12x-3=0$; (*A*).

One whose roots are 2 less than those of (*A*),

$$x^4+8x^3+12x^2-4x-11=0; \quad (B).$$

One whose roots are .8 less than those of (*B*), or 2.8 than those of (*A*),

$$x^4+11.2x^3+35.04x^2+32.608x-2.0144=0; \quad (C).$$

One whose roots are .05 less than those of (*C*), .85 less than those of (*B*), or 2.85 less than those of (*A*),

$$x^4+11.4x^3+36.735x^2+36.1965x-.29499375=0.$$

8. Transform, as above, the equation $x^3-7x+7=0$, successively, into equations whose roots shall be 1 less, 1.3 less, 1.35 less, and 1.356 less than the roots of the given equation.

271. Prop.—*If $a+x_1$ is a root of $f(x)=0$, and x_1 is sufficiently small with reference to a, $x_1=-\frac{f(a)}{f'(a)}$, approximately.*

DEM.—If $a+x_1$ is a root of $f(x)=0$, $f(a+x_1)=0$. Developing this by Taylor's Formula, we have

$$f(a+x_1)=f(a)+f'(a)x_1+f''(a)\frac{x_1^2}{2}+f'''(a)\frac{x_1^3}{\lfloor 3}+\text{ etc.}=0.$$

Now, to determine x_1 *approximately*, which is all the proposition proposes, when x_1 is quite small with reference to a, all the terms in the development involving higher powers of x_1 than the first may be neglected; whence we have $f(a)+f'(a)x_1=0$, or $x_1=-\frac{f(a)}{f'(a)}$.

EX.—Knowing that 4.+ some decimal fraction which we will call x_1 is a root of $x^3+x^2+x-100=0$, required the approximate value of the decimal fraction x_1.

SOLUTION.—Finding $f(a)$, *i. e.*, in this case $f(4)$* in the ordinary way, we have

1	+1	+1	−100	4
	4	20	84	
	5	21	$-16=f(a)$, or $f(4)$*	
	4	36		
	9	$57=f'(a)$, or $f'(4)$*		
	4			
	13			

Hence $-\frac{f(a)}{f'(a)}=-\frac{-16}{57}=.28+$ is *approximately* the decimal part of the root. *In fact*, 2 *is* the tenths figure of the decimal part of the root, the root being (as we shall find hereafter) 4.2644+.

We thus have $x_1^3+13x_1^2+57x_1-16=0$, an equation whose roots are 4 less than the roots of the given equation. We will now transform this into another equation whose roots shall be .2 less than the roots of this equation, or 4.2 less than the roots of the *given* equation. Thus

1	+13	+57	−16	.2
	.2	2.64	11.928	
	13.2	59.64	$-4.072=f(4.2)$†	
	.2	2.68		
	13.4	$62.32=f'(4.2)$†		
	2			
	13.6			

and the transformed equation is

$$x_2^3+13.6x_2^2+62.32x_2-4.072=0,$$

* This notation means, the value of $f(x)$ when 4 is substituted for x therein.

† That these are the values of $f(x)$ (the first member of the given equation) and $f'(x)$, when 4.2 is substituted for x, will be evident if it is considered that they are the same results as would have been obtained by transforming the given equation immediately (by one process) into another whose roots are 4.2 less.

which is an equation whose roots are 4.2 less than those of the given equation, *i. e.*, $x = 4.2 + x_2$.

Hence by the proposition $x_2 = -\frac{-4.072}{62.32} = .065$ approximately. In fact, it will be seen that 6 *is* the hundredths figure of the root.

Writing both portions of the above work together, it stands thus:

1	+1	+1	−100	| 4.2	
	4	20	84		
	5	21	−16*		* ∴ $-16 = f(a)$, or $f(4)$
	4	36	11.928		
	9	57*	−4.072†		* ∴ $57 = f'(a)$, or $f'(4)$
	4	2.64			
	13*	59.64			† ∴ $-4.072 = f(4.2)$
	.2	2.68			
	13.2	62.32†			† ∴ $62.32 = f'(4.2)$
	.2				
	13.4				
	.2				
	13.6†				

Horner's Rule.

272. *RULE.*—1. PUT THE EQUATION IN THE FORM

$$Ax^n + Bx^{n-1} + Cx^{n-2} - - - - Mx + L = 0,$$

IN WHICH THE COEFFICIENTS A, B, C - - - - L, IF NOT INTEGRAL, ARE EXPRESSED EXACTLY IN DECIMAL FRACTIONS.

2. FIND THE NUMBER AND SITUATION OF THE POSITIVE REAL ROOTS BY STURM'S THEOREM, DETERMINING ONE OR MORE (USUALLY TWO) OF THE INITIAL FIGURES. (See SCH. 1.)

3. WRITE THE COEFFICIENTS IN ORDER WITH THEIR PROPER SIGNS, BEING CAREFUL TO SUPPLY WITH 0'S THE PLACES OF COEFFICIENTS OF MISSING TERMS, IF THE EQUATION IS NOT COMPLETE. TAKING THE INITIAL FIGURES OF ONE OF THESE ROOTS AS THUS FOUND, OPERATE ON THESE COEFFICIENTS SO AS TO OBTAIN THE COEFFICIENTS OF THE TRANSFORMED EQUATION WHOSE ROOTS SHALL BE LESS BY THE PORTION OF THIS ROOT ALREADY FOUND.

4. HAVING FOUND THESE COEFFICIENTS, IF THE COEFFICIENT OF THE FIRST POWER OF THE UNKNOWN QUANTITY IN THIS TRANS-

FORMED EQUATION AND THE ABSOLUTE TERM, $f'(a)$ AND $f(a)$, HAVE UNLIKE SIGNS, DIVIDE THE LATTER BY THE FORMER, AND THE FIRST FIGURE OF THIS QUOTIENT WILL BE (APPROXIMATELY) THE NEXT FIGURE OF THE ROOT. (See SCH. 2.) IF THESE FUNCTIONS HAVE LIKE SIGNS, MORE FIGURES OF THE ROOT MUST BE FOUND BY STURM'S THEOREM OR BY TRIAL, BEFORE PROCEEDING TO APPLY THIS PROCESS OF TRANSFORMATION.

5. HAVING FOUND A FIGURE OF THE ROOT BY DIVIDING $f(a)$ BY $f'(a)$, ANNEX IT TO THE ROOT AND OPERATE ON THE COEFFICIENTS OF THE LAST (TRANSFORMED) EQUATION AS THEY STAND, TO PRODUCE THE COEFFICIENTS OF THE NEXT TRANSFORMED EQUATION, *i. e.*, THE ONE WHOSE ROOTS SHALL BE LESS THAN THOSE OF THE LAST, BY THE LAST FIGURE OF THE ROOT, AND LESS THAN THOSE OF THE GIVEN EQUATION BY THE ENTIRE PORTION OF THE ROOT NOW FOUND. HAVING FOUND THESE COEFFICIENTS, DIVIDE THE ABSOLUTE TERM BY THE COEFFICIENT OF THE FIRST POWER OF THE UNKNOWN QUANTITY, IF THEIR SIGNS ARE UNLIKE, AND THE FIRST FIGURE OF THIS QUOTIENT WILL BE (APPROXIMATELY) THE NEXT FIGURE OF THE ROOT. IF THESE SIGNS ARE ALIKE, THE LAST ASSUMED FIGURE OF THE ROOT IS TOO LARGE AND MUST BE DIMINISHED. (See SCH. 3.)

6. PROCEED IN THIS MANNER UNTIL THE ROOT IS OBTAINED; OR, IF THE ROOT IS INCOMMENSURABLE, UNTIL AS MANY FIGURES OF THE DECIMAL FRACTION ARE OBTAINED AS ARE DESIRED. (See SCH. 4.)

7. IN LIKE MANNER ALL THE POSITIVE REAL ROOTS, OR THEIR APPROXIMATE VALUES, MAY BE FOUND. TO OBTAIN THE NEGATIVE ROOTS, CHANGE THE SIGNS OF ALL THE TERMS CONTAINING ODD POWERS OF THE UNKNOWN QUANTITY, OR ALL OF THOSE CONTAINING THE EVEN POWERS; OR, IF THE EQUATION IS COMPLETE, EACH ALTERNATE SIGN, AND PROCEED TO FIND THE POSITIVE ROOTS OF THIS EQUATION AS BEFORE. THE VALUES THUS FOUND WILL BE THE NUMERICAL VALUES OF THE NEGATIVE ROOTS (***246***).

This rule is based upon previously demonstrated principles, and needs no special demonstration.

273. SCH. 1.—By means of (***244, 245***) we can usually find the initial figure or figures of the roots with less labor than by Sturm's Theorem.

274. SCH. 2.—Since by (**271**) $x_1 = -\frac{f(a)}{f'(a)}$, if both $f(a)$ and $f'(a)$ have the same sign *at any time*, this quotient will be −, and hence the value thus found for x_1 will not be the amount to be *added* (annexed) to the portion of the root already found, for the assumption is that this portion is *less* than the root of the equation which we are seeking.

275. SCH. 3.—That the figure of the root found by dividing $f(a)$ by $f'(a)$ is liable to be too large is readily seen when we consider that instead of $f'(a)x_1 = -f(a)$ (in DEM. of **271**), we should have, if no terms were omitted,

$$f'(a)x_1 + \tfrac{1}{2}f''(a)x_1^2 + \tfrac{1}{6}f'''(a)x_1^3 + \text{etc.} = -f(a).$$

Now a value of x_1 which satisfies the former may evidently be quite too large to satisfy the latter. Thus consider $x^3+10x^2+5x-2600=0$. Neglecting x^3 and $10x^2$, we have $5x=2600$, or $x=520$. But this will by no means satisfy the equation when x^3 and $10x^2$ are not neglected.

Again, the figure found by dividing $f(a)$ by $f'(a)$ may be too small. Thus, if we have $x^4-12x^2+12x-3=0$, and neglect x^4, and $-12x^2$, we have $12x-3=0$, or $x=\frac{1}{4}$. But this is too small a value to satisfy the equation, since for $x=\frac{1}{4}$, $-12x^2$ will be numerically much larger than x^4, and hence retaining these terms will diminish the function, thus making $\frac{1}{4}$ too small to satisfy the equation.

276. SCH. 4.—From SCH. 2 it appears that $f(a)$ cannot change sign in the process unless $f'(a)$ also changes sign. But when $f(a)$ changes sign, we know by (**244**) that we have passed a root of the equation; if, however, $f'(a)$ also changes at the same time, our work may still be right. In such a case there are two roots having their initial figure or figures alike, *e. g.*, one may be 23.56+, and the other, 23.59+. To obtain the less of the two roots, take the *largest* figure which will not cause either $f(a)$ or $f'(a)$ to change sign; and for the larger of the two roots take the *smallest* figure which will cause both $f(a)$ and $f'(a)$ to change sign.

[NOTE.—These scholiums, as also the rule, will be better understood in connection with their applications in the following examples. But in review, after the solution of the examples, they should be carefully learned.]

EXAMPLES.

1. Required the roots of $x^3-4x^2-6x+8=0$.

SOLUTION.—By Sturm's method we find that there are 3 real roots, one negative, and two positive (see Ex. 1, page 223), and also that the negative root is −1. and an incommensurable decimal, that one positive root is an incommensurable decimal, and that the other positive root is 4. + an incommensurable decimal. We will seek the latter first.

OPERATION.

1	−4	−6	+ 8	4.892+
	4	0	−24	
	0	−6	−16.···	
	4	16	13.632	
	4	10 ··	−2.368···	
	4	7.04	2.309769	
	8.8	17.04	−.058231···	
	.8	7.68	.053275288	
	9.6	24.72··	−.004955712	
	.8	.9441		
	10.49	25.6641		
	.09	.9522		
	10.58	26.6163··		
	.09	.021344		
	10.672	26.637644		
	.002	.021348		
	10.674	26.658992		
	.002			
	10.676			

REMARKS.—The general features of the process, being the same as heretofore given (***270,*** Example), need no further explanation than they have already received. Each decimal figure of the root is added the first time in the first column simply by annexing it.

In finding the second figure of the root, we have $-\frac{f(a)}{f'(a)} = -\frac{-16}{10} = 1.6$. But this cannot be the proper addition, since we know that the root lies *between* 4 and 5; hence this trial fails to give the second figure in the root. (See ***275.***) But as we know that this figure cannot be greater than 9, we try 9, and find that it makes the absolute term change sign so that $f(a)$ and $f'(a)$ have the same sign, and consequently .9 is too much to add. (See ***276,*** and also consider that $f(x)$ would thus be shown to change sign as x passed from 4 to 4.9, and hence that a root lies between 4 and 4.9, ***244.***) We therefore try .8, and find that it is the correct addition. We *know* that .8 is right, since we know that as x passes from 4.8 to 4.9, $f(x)$ changes sign.

In finding the third figure we have for trial $-\frac{f(a)}{f'(a)} = -\frac{-2.368}{24.72} = .09$. Trying 9 as the third figure of the root, we find that the absolute term does not change sign, and hence we *know* that 9 is the next figure, *i. e.*, we know that a root lies between 4.89 and 4.9.

The process may be thus continued indefinitely, and as many figures found as we may desire.

277. N. B.—It will be observed that this process is simply one of substitution in $f(x)$ of values for x which come nearer and nearer to making $f(x) = 0$.

Thus in this example 4, substituted in $x^3 - 4x^2 - 6x + 8$, gives $x^3 - 4x^2 - 6x + 8 = -16$; 4.8 substituted for x, gives $x^3 - 4x^2 - 6x + 8 = -2.368$; 4.89 gives $x^3 - 4x^2 - 6x + 8 = -.058231$; 4.892 gives $x^3 - 4x^2 - 6x + 8 = -.004955712$. Thus we are coming nearer and nearer to the number which substituted for x would make $x^3 - 4x^2 - 6x + 8 = 0$, or would satisfy the equation.

2. To FIND THE ROOT WHICH LIES BETWEEN −1 AND −2, we take the equation $x^3 + 4x^2 - 6x - 8 = 0$ (changing the signs of the terms containing the even powers of x), and find the root of this equation which lies between 1 and 2 (***246***).

OPERATION.

1	+4	−6	−8	1.8004+
	1	5	−1	
	5	−1	−9 ···	
	1	6	8.992	
	6	5.··	−.008··········	
	1	6.24	.007249504064	
	7.8	11.24	−.000750495936	
	.8	6.88		
	8.6	18.12······		
	.8	.00376016		
	9.4004	18.12376016		
	.0004	.00376032		
	9.4008	18.12752048		
	.0004			
	9.4012			

3. To FIND THE ROOT WHICH LIES BETWEEN 0 AND 1. We first find the initial figure either by evaluating $f(x)$ successively for .1, .2, .3, etc., and noticing when it changes sign (***244***); or by Sturm's method. The former is much the less laborious, and is to be preferred (***265***). In fact, to use Sturm's method involves *exactly the same work as the former method*, with considerable additional work. Moreover, the former method can be applied mentally till the proper initial figure is determined, and no other writing will need to be involved than just what Horner's method requires. No figures will need to be written but those in the following

OPERATION.

1	−4.·	−6.··	+8.···	.9082+
	.9	−2.79	−7.911	
	−3.1	−8.79	.089······	
	.9	−1.98	−.086242688	
	−2.2	−10.77····	.002757312	
	.9	.010336		
	−1.3··	−10.780336		
	.008			
	−1.292			

It is so evident that the last figure is 2, that the operation for verifying it is unnecessary.

2. Find the roots of $x^5 - 13x^4 + 53x^3 - 49x^2 - 110x + 150 = 0$, extending the decimals to the 5th place.

SUG.—Apply Sturm's method. If there are equal roots, depress the equation.

3 to 5. Find all the real roots of the following, extending the decimals to 4 or 5 places:

(3.) $x^3 + 10x^2 + 5x - 260 = 0$;
(4.) $x^3 + \ 3x^2 + 5x = 178$;
(5.) $x^3 + \ 2x^2 = 23x + 70$.

The cubic equations on pages 223, 224, 226, 228, will afford further exercise.

6. Find the roots of the equation $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$.

SUG'S.—Of course we may always find the number and situation of the real roots by Sturm's method. But as the labor of substituting in *all* the functions used in this method is frequently great, we avoid it when we can. *However, it is generally best to free the equation from equal roots, and find the* NUMBER *of positive, and the* NUMBER *of negative roots by Sturm's method.* But the *situation* of the roots is almost always more readily found by inspection based mainly on the change in sign of $f(x)$ (**244**). We will solve this example in this way.

1. By Sturm's method we find that our equation has no equal roots, and that it has 4 positive roots, and no negative root (see **264**).

2. WE NOW PROCEED TO FIND THE LEAST ROOT. Observing that for $x=0$, $f(x)$ is $+$, and for $x=1$, $f(x)$ is $-$, we know that at least one real root lies between these limits. To find it we have the following (see next page):

FIRST OPERATION.

1	−80	+1998	−14937	+5000	.1
	.1	− 7.99	199.001	−1473.7999	.2
	−79.9	1990.01	−14737.999	3526.2001*	.3
	.1	− 7.98	198.203	−2829.6320	.02
	−79.8	1982.03	−14539.796*	696.5681····†	.02
	.1	− 7.97	391.636	− 274.42385424	.01
	−79.7	1974.06*	−14148.160	422.14424576‡	.35
	.1	− 15.88	388.468	− 272.88640240	
	−79.6*	1958.18	−13759.692···†	149.25784336§	
	.2	− 15.84	38.499288	− 135.86783711	
	−79.4	1942.34	−13721.192712	13.39000625¶	
	.2	− 15.80	38.467784		
	−79.2	1926.54··†	−13682.724928‡		
	.2	− 1.5756	38.404808		
	−79.0	1924.9644	−13644.320120		
	.2	− 1.5752	38.373336		
	−78.8·†	1923.3892	−13605.946784§		
	.02	− 1.5748	19.163073		
	−78.78	1921.8144‡	−13586.783711		
	.02	− 1.5740	19.155211		
	−78.76	1920.2401	−13567.628500¶		
	.02	− 1.5736			
	−78.74	1918.6668			
	.02	− 1.5732			
	−78.72‡	1917.0936§			
	.02	− .7863			
	−78.70	1916.3073			
	.02	− .7862			
	−78.68	1915.5211			
	.02	− .7861			
	−78.66	1914.7350¶			
	.02				
	−78.64§				
	.01				
	−78.63				
	.01				
	−78.62				
	.01				
	−78.61				
	.01				
	−78.60				

REMARKS.—This work is given to show how we *may* proceed to find the first two figures of the root by successive simple approximations. If the student is familiar with the principles heretofore developed and applied, he will have no difficulty in seeing the reasons for the operations above. We are simply adding to the value of x substituted in $f(x)$, so as steadily to diminish the absolute term, being careful not to add so great an amount to x as to make this term change its sign; and when we can add no more of one order (as of *tenths*), we pass to the next lower order (hundredths) and proceed in the same manner. On this process we make two remarks, viz.:

(*a.*) *It is not sure to succeed.* Thus, if there were *two* roots between .34 and .35, for example, the absolute term would not change sign when we passed from .34 to .35, although we would have passed both roots; and it might occur that no root lay beyond .35, in which case our method would be fruitless. But such cases are rare. It is in such cases, and in such only, that Sturm's method is well-nigh indispensable for finding the *situation* of roots.

(*b.*) In most cases the *exact* figure of any order can be told without such an approximation as the above; or, what is equivalent, without trying a figure, and when it is found incorrect, erasing the work and trying another, and so on till the right figure is found. In this particular case, *the first figure in the root being a small fraction*, the higher powers of x might be neglected (and more especially as they differ in signs), and $-14937x + 5000 = 0$ would give the first figure in the root at once. Thus $x = \frac{5000}{14937} = .3+$. So, *in this case*, for the second figure $-\frac{f(a)}{f'(a)} = -\frac{696.5681}{-13759.692} = .05+$, which gives the next figure of the root.

3. TO FIND THE NEXT GREATER ROOT. By substituting 1, we find, as on the next page, $f(x) = -8018$; and when 1 is added to this, $f(x) = -17506$. Now it is evident that any *slight* addition, as of 2, 3, or 4, to the value of x, will only make $f(x)$ increase negatively. This is seen by inspecting the coefficients 1, -72, $+1542$, -7873, -17506. We therefore make a considerably larger addition to x, as 10. From this explanation the student should be able to see the significance of the following (see next page:)

SECOND OPERATION.

1	−80	+1998	−14937	+ 5000	1
	1	− 79	1919	−13018	1
	−79	1919	−13018	− 8018	10
	1	− 78	1841	− 9488	12.7
	−78	1841	−11177	−17506	
	1	− 77	1689	13470	
	−77	1764	− 9488	− 4036	
	1	− 75	1615	3737.3441	
	−76	1689	− 7873	− 298.6559	
	1	− 74	9220		
	−75	1615	1347		
	1	− 73	4020		
	−74	1542	5367 ...		
	1	− 620	− 27.937		
	−73	922	5339.063		
	1	− 520	− 42.931		
	−72	402	5296.132		
	10	− 420			
	−62	− 18 ..			
	10	− 21.91			
	−52	− 39.91			
	10	− 21.42			
	−42	− 61.33			
	10	− 20.93			
	−32 .	− 82.26			
	.7				
	−31.3				
	.7				
	−30.6				
	.7				
	−29.9				
	.7				
	−29.2				

As now $f(a)$ and $f'(a)$ have opposite signs, and the remainder of the root is quite small as compared with that already found, the approximation can be carried on in the ordinary way. Thus we have $-\frac{f(a)}{f'(a)} = -\frac{-298.6559}{5296.132} = .05+$, and the next figure of the root is 5.

4. To FIND THE NEXT GREATER ROOT we resume the coefficients after the roots had been diminished by 12. Then adding 1 to the value of x, we find that

for $x=13$, $f(x)=1282$, having changed sign, as it should. Now as $f'(x)$, *i. e.* 5239, and $f(x)$ are both positive, and the other coefficients, though negative, are comparatively small, it will take considerable increase in x to change the sign of $f(x)$. We therefore add 10. Now $f'(x)$ has changed sign, and by inspecting the coefficients, 1, +12, −348, −1321, and 24872, it is evident that x cannot increase another 10 without changing the sign of $f(x)$. Hence we try 5. For a similar reason we add 4 next.

THIRD OPERATION.

1	−32	− 18	+5367	− 4036	12
	1	− 31	− 49	5318	1
	−31	− 49	5318	1282*	10
	1	− 30	− 79	23590	5
	−30	− 79	5239*	24872†	4
	1	− 29	−2880	−13180	32.+
	−29	−108*	2359	11692‡	
	1	−180	−3680	−11588	
	−28*	−288	−1321†	104§	
	10	−80	−1315		
	−18	−368	−2636		
	10	20	− 765		
	− 8	−348†	−3401‡		
	10	85	504		
	2	−263	−2897		
	10	110	1144		
	12†	−153	−1753§		
	5	135			
	17	− 18‡			
	5	144			
	22	126			
	5	160			
	27	286			
	5	176			
	32‡	462§			
	4				
	36				
	4				
	40				
	4				
	44				
	4				
	48§				

Now $-\dfrac{f(a)}{f'(a)}=-\dfrac{104}{-1753}=.05+$. But as the coefficients preceding −1753 are all +, they will diminish it somewhat in the operation, and hence it is probable that .06 is the proper addition to make to the root. The process can now be continued to any extent desired.

5. TO FIND THE NEXT GREATEST (in this case the greatest) ROOT, we have the following operation, which we leave the student to trace:

FOURTH OPERATION.

					32
1	+48	+462	−1753	+ 104	1
	1	49	511	−1242	1
	49	511	−1242	−1138*	34.8 +
	1	50	561	16	
	50	561	− 681*	−1154.····	
	1	51	665	1086.8416	
	51	612*	− 16	− 67.1584	
	1	53	719		
	52*	665	703.···		
	1	54	655.552		
	53	719	1358.552		
	1	55	692.416		
	54	774.··	2050.968		
	1	45.44			
	55	819.44			
	1	46.08			
	56.8	865.52			
	.8	46.72			
	57.6	912.24			
	.8				
	58.4				
	.8				
	59.2				

The student should extend these solutions 2 or 3 figures farther.

7 to 12. Solve the following:

(7.) $x^3 + 60x^2 - 800x = 60000.$

(8.) $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321.$

(9.) $x^4 + 4x^3 - 4x^2 - 11x + 4 = 0.$

(10.) $x^4 - 27x^3 + 162x^2 + 356x = 1200.$

(11.) $x^3 - 3x^2 = 48654231721.$

(12.) $x^3 + 2x^2 + 3x = 13089030.$

(13.) $x^5 - 10x^3 + 6x = 1.$

(14.) $x^3 + 173x = 14760638046.$

(15.) $x^3 - 7035x^2 + 15262754x = 10000730880$.

(16.) $x^2 + 12x = 35.4025$. (Solve by Horner's method.)

(17.) $x^3 + 4x^2 - 9x = 57.623625$.

(18.) $2x^4 + 5x^3 + 4x^2 + 3x = 8002$. (Observe that it is not *necessary* to make the first coefficient unity. See examples in the first part of the section.)

(19.) $3x^4 - 4x^2 + 2x = 1000$.

(20.) $5x^3 - 3.2x = 41278.216$.

NOTE.—The roots of several of the above are commensurable; and their solution shows that Horner's method is adapted to such cases.

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21 to 25. Extract the roots of the following numbers by Horner's method:

(21.) The cube root of 119736852154.

(22.) The square root of 5126485.

(23.) The fifth root of 2.

(24.) The fourth root of 3571827100256 7691.

(25.) The cube root of 3.

SUG'S.—To solve the 21st, write $x^3 - 119736852154$, and solve as usual, being careful to remember that the coefficients are 1, 0, 0, −119736852154. To find the initial figure, point off as in the ordinary method of extracting roots. The following exhibits the first steps of the process:

1	0	0	−119736852154	49
	4	16	64	
	4	16	− 55736	
	4	32	53649	
	8	48··	− 2087852	
	4	1161		
	129	5961		
	9	1242		
	138	7203		
	9			
	147			

---

26 to 29. Solve the following by first eliminating, and then solving the resulting equation by Horner's method:

(26.) $2x^2 - 5x + 3y = 2xy - 4x^2 + 12$, and $4y^2 - 3x = 2y + 5$.

(27.) $2y^2 - 4xy + 2x^2 - 3y - 2x - 8 = 0$, and $4y^2 + 4x^2 = 11$.

(28.) $2y^2 - 4xy + 2x^2 - 3y - 2x = 8$, and $y^2 \times 2y + x^2 - 6x = -6$.

(29.) $2y^2 - 4xy + 2x^2 - 3y - 2x = 8$, and $y^2 + 6y + x^2 - 4x + 9 = 0$.

SUG'S.—From the 2d of (26) we have $y = \frac{1}{4} \pm \frac{1}{2}\sqrt{3x + \frac{21}{4}}$. Substituting this in the 1st, we obtain $6x^2 - \frac{11}{2}x - \frac{45}{4} = (x - \frac{3}{2})\sqrt{3x + \frac{21}{4}}$, whence $36x^4 - 69x^3 - 101x^2 + \frac{531}{4}x + \frac{459}{4} = 0$. And dividing by 36, we have $x^4 - 1.917x^3 - 2.806x^2 + 3.687x + 3.188 = 0$, carrying the fractions to three places.

278. SCH.—There are various methods by which Horner's process may be abridged, especially when a large number of decimals is required; but we have thought it better to exhibit fully and clearly the principles *essential* to the process, than to spend time and distract attention by giving these arithmetical abridgments. The most simple of these are: (*a*) the omission of the decimal point; (*b*) the writing of the *sums* only in the several working columns, performing the various multiplications and additions mentally; (*c*) after several decimals have been obtained, instead of annexing 0's (or ··· 's) to the working columns, *dropping off* figures from *the right* in each new operation, as one from next to the last right-hand column, two from the next to the left, three from the next to the left, etc.; (*d*) and, finally, when all the working columns but the last two have disappeared, continuing the operation as a process of simple division, only dropping off a figure from the right of the divisor at each step instead of annexing a 0 to the dividend. We condense an example from Todhunter as an illustration.

Ex.—To compute to 16 decimal places the root of $x^3 + 3x^2 - 2x - 5 = 0$, which lies between 1 and 2.

OPERATION.

1	+3	−2	−5	1.3300587395679821
	4	2	−3000	
	5	700	−333000	
	60	889	−663000000000	
	63	108700	−98647524875	
	66	110779	−8347885443	
	690	112867000000	−446624425	
	693	112870495025	−107998801	
	696	112873990075	−6411112	
	699000	11287454929	−767351	
	699005	11287510850	−90100	
	699010	1128751574	−11087	
	699015	1128752063	−929	
		112875208	−27	
		112875210	−4	

SECTION III.

GENERAL SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS.

Cardan's Solution of Cubic Equations.

279. Prob.—*To resolve the general cubic equation* $x^3 + px^2 + qx + r = 0$.

Solution.—This solution consists of three steps: 1. To transform the equation into one of the form $y^3 + my + n = 0$, that is, an incomplete cubic lacking the square of the unknown quantity. To effect this, we put $x = y + z$, and substituting, have

$$y^3 + 3y^2z + 3yz^2 + z^3 + py^2 + 2pyz + pz^2 + qy + qz + r = 0,$$
$$\text{or, } y^3 + (3z + p)y^2 + (3z^2 + 2pz + q)y + z^3 + pz^2 + qz + r = 0. \qquad (1)$$

Now as we have only one condition expressed between y and z, viz., $y+z=x$, we are at liberty to impose another. Let us put $3z + p = 0$, whence $z = -\frac{1}{3}p$. Then will this value of z substituted in (1) give

$$y^3 + (q - \tfrac{1}{3}p^2)y + (\tfrac{2}{27}p^3 - \tfrac{1}{3}pq + r) = 0. \qquad (2)$$

2. Since the above transformation can always be effected, a solution of

$$y^3 + my + n = 0 \qquad (3)$$

will include the solution of all cubic equations. Our second step is to transform this equation into one which can be solved as a quadratic. To do this we put $y = u + v$, which gives (3) the form

$$u^3 + 3u^2v + 3uv^2 + v^3 + m(u + v) + n = 0,$$
$$\text{or, } u^3 + 3uv(u + v) + v^3 + m(u + v) + n = 0,$$
$$\text{or, } u^3 + v^3 + (3uv + m)(u + v) + n = 0. \qquad (4)$$

Now, as we have but one condition expressed between u and v, viz., $u+v=y$, we are at liberty to impose another. Let us put $3uv + m = 0$, whence $v = -\frac{m}{3u}$; and (4) becomes $\quad u^3 + v^3 + n = 0,$

or by substituting the value of v,

$$u^3 - \frac{m^3}{27u^3} + n = 0,$$

whence we have $\quad u^6 + nu^3 = \frac{1}{27}m^3. \qquad (5)$

3. Solving this quadratic we obtain

$$u^3 = -\tfrac{1}{2}n \pm \sqrt{\tfrac{1}{27}m^3 + \tfrac{1}{4}n^2}, \text{ or } u = \sqrt[3]{-\tfrac{1}{2}n \pm \sqrt{\tfrac{1}{27}m^3 + \tfrac{1}{4}n^2}};$$

and as $v^3 = -(u^3 + n)$, $\quad v = \sqrt[3]{-\frac{1}{2}n \mp \sqrt{\frac{1}{27}m^3 + \frac{1}{4}n^2}}.$

Finally, taking the square root as + for the value of u, and − for the value of v, since these are corresponding values, we have

$$y = \sqrt[3]{-\tfrac{1}{2}n + \sqrt{\tfrac{1}{27}m^3 + \tfrac{1}{4}n^2}} + \sqrt[3]{-\tfrac{1}{2}n - \sqrt{\tfrac{1}{27}m^3 + \tfrac{1}{4}n^2}}. \qquad (6)$$

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280. Prop.—1. *In the equation* $y^3 + my + n = 0$, *when* m *is positive, and when* m *is negative and* $\frac{1}{27}m^3 < \frac{1}{4}n^2$, *the equation has one real and two imaginary roots, and Cardan's formula* (6) *gives a satisfactory solution.*

2. *When* m *is negative and* $\frac{1}{27}m^3 = \frac{1}{4}n^2$, *two of the roots are equal, and Cardan's method is satisfactory.**

3. *But, when* m *is negative and* $\frac{1}{27}m^3 > \frac{1}{4}n^2$, *all the roots are real and unequal, while Cardan's method makes them apparently imaginary, and the solution is unsatisfactory.*

DEM.—A cubic equation must have at least one *real* root (***238***). Let this be a. Now conceive the equation reduced to a quadratic by dividing $f(x)$ by $x-a$, and let $b + \sqrt{c}$, and $b - \sqrt{c}$ be the roots of this quadratic, these being the general forms of the roots of a quadratic, in which if c is + the roots are real, if c is − they are imaginary, and if c is 0 these two roots are equal.

Now, a, $b + \sqrt{c}$, and $b - \sqrt{c}$ being the roots of the equation, we have by (***235***)

$$(x-a)(x-[b+\sqrt{c}])(x-[b-\sqrt{c}]) = x^3-(a+2b)x^2+(2ab+b^2-c)x-a(b^2-c)=0.$$

To transform this into the form $y^3 + my + n = 0$, we must put $a + 2b = 0$; whence $a = -2b$, and we have

$$y^3 - (3b^2 + c)y + 2b(b^2 - c) = 0.$$

Comparing this with Cardan's formula, we see that

$$\sqrt{\tfrac{1}{27}m^3 + \tfrac{1}{4}n^2} = \sqrt{-\tfrac{1}{27}(3b^2 + c)^3 + b^2(b^2 - c)^2} = \sqrt{-3b^4c + \tfrac{2}{3}b^2c^2 - \tfrac{1}{27}c^3}$$
$$= \sqrt{(b^4 - \tfrac{2}{9}b^2c + \tfrac{1}{81}c^2)(-3c)} = (b^2 - \tfrac{1}{9}c)\sqrt{-3c}.$$

Hence we see that if c is +, that is, if all the roots of a cubic equation are *real and unequal*, Cardan's method gives a result apparently imaginary. But if c is −, that is, if two of the roots are imaginary, Cardan's method gives a *real* form. Also when $c = 0$, that is, when the roots are a, b, and b, the form is *real*, since $\sqrt{\frac{1}{27}m^3 + \frac{1}{4}n^2} = (b^2 - \frac{1}{9}c)\sqrt{-3c}$, is then 0.

Now by inspecting the quantity $\sqrt{\frac{1}{27}m^3 + \frac{1}{4}n^2}$ we see that it is *real* when m is positive; and also when m is negative if $\frac{1}{27}m^3 < \frac{1}{4}n^2$. Hence in these

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* If *all* the roots are equal, the equation takes the form $(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3 = 0$, a being the value of one of the equal roots (***235***). In this case the transformation which makes the term in x^2 disappear gives $y^3 = 0$, since $x = y - \frac{1}{3}p = y + a$, and $y = x - a = 0$.

cases there are one *real* and two imaginary roots, and Cardan's method, giving a real form, enables us to determine one of them, and hence to solve the equation.

2d. We have also seen above that when $c = 0$, that is, when two of the roots are equal (and not all three), $\sqrt{\frac{1}{27}m^3 + \frac{1}{4}n^2} = 0$, in which case m must be negative and $\frac{1}{27}m^3 = \frac{1}{4}n^2$.

3d. It has also appeared above that when all the roots are ***real and unequal***, Cardan's method gives an apparently imaginary result. But this can only be the case when m is negative, and $\frac{1}{27}m^3 > \frac{1}{4}n^2$.

281. Sch.—Cardan's method would seem to give a cubic equation *nine* roots instead of three, since as there are three cube roots of any number, $\sqrt[3]{-\frac{1}{2}n + \sqrt{\frac{1}{27}m^3 + \frac{1}{4}n^2}}$ would have three values, and $\sqrt[3]{-\frac{1}{2}n - \sqrt{\frac{1}{27}m^3 + \frac{1}{4}n^2}}$ would have three other values. Now combining each of the former, in turn, with each of the latter, we should have *nine* results. In order to explain this seeming paradox, let us find the form of the three cube roots of a number, as of a^3. To do this we have but to solve the equation $x^3 = a^3$. Thus $x^3 - a^3 = (x-a)(x^2 + ax + a^2) = 0$. Whence $x - a = 0$, and $x^2 + ax + a^2 = 0$. From these we have $x = a$, $-\frac{1}{2}a(1 + \sqrt{-3})$, and $-\frac{1}{2}a(1 - \sqrt{-3})$. Now let the roots of $\sqrt[3]{-\frac{1}{2}n + \sqrt{\frac{1}{27}m^3 + \frac{1}{4}n^2}}$ be r, $-\frac{1}{2}r(1 + \sqrt{-3})$, and $-\frac{1}{2}r(1 - \sqrt{-3})$; and the roots of $\sqrt[3]{-\frac{1}{2}n - \sqrt{\frac{1}{27}m^3 + \frac{1}{4}n^2}}$ be r', $-\frac{1}{2}r'(1 + \sqrt{-3})$, and $-\frac{1}{2}r'(1 - \sqrt{-3})$. It will be remembered that we assumed $uv = -\frac{m}{3}$; that is, the products of the admissible roots must be *real*. Therefore we can use for the parts of the root r and r', $-\frac{1}{2}r(1 + \sqrt{-3})$ and $-\frac{1}{2}r'(1 - \sqrt{-3})$, and $-\frac{1}{2}r(1 - \sqrt{-3})$ and $-\frac{1}{2}r'(1 + \sqrt{-3})$; and we can use these parts in no other combination, as any other would not give a real quantity. Thus we cannot have $y = u + v = r - \frac{1}{2}r(1 + \sqrt{-3})$, since uv would then be $-r[\frac{1}{2}r(1 + \sqrt{-3})]$, which is an imaginary quantity, and hence not equal to $-\frac{m}{3}$, as it should be.

We will give a few examples to which the student may apply Cardan's process.

Examples.

Solve the following, finding one of the roots by Cardan's process, and then depressing the equation by division, solve the resulting quadratic.

1. $x^3 - 9x + 28 = 0$.*
2. $x^3 - 3x^2 + 4 = 0$. (See first step in general solution.)
3. $x^3 - 6x + 4 = 0$.
4. $x^3 + 6x - 2 = 0$.
5. $x + b + 3\sqrt[3]{abx} = a$.
6. $x^3 + 3x^2 + 9x - 13 = 0$.
7. $x^3 - 9x^2 + 6x - 2 = 0$.
8. $x^3 - 6x^2 + 13x - 10 = 0$.
9. $x^3 - 48x = 128$.
10. $x^3 + 2x = 12$.
11. $z^6 - 3z^4 - 2z^2 - 8 = 0$.†
12. $y^3 - 6y^2 + 13y = 12$.
13. $2x^3 - 12x^2 + 36x = 44$.
14. $\dfrac{\sqrt{a+x}}{a} + \dfrac{\sqrt{a+x}}{x} = \dfrac{\sqrt{x}}{c}$.
15. $x^3 - 8x^2 + 19x - 12 = 0$.

SUG.—An attempt to solve the last by Cardan's process will give roots *apparently* imaginary, although it is easy to see that the roots are all real, and commensurable.

DESCARTES'S SOLUTION OF BIQUADRATICS.

282. Prob.—*To resolve the general biquadratic equation* $x^4 + ax^3 + bx^2 + dx + e = 0$.

SOLUTION.—The first step in the process is to transform the equation into one wanting the 3d power of the unknown quantity. This is done in the usual way (see Cardan's method of resolving cubics); *i. e.*, by putting $x = y + z$, substituting, collecting the coefficients with reference to y, and, putting the coefficient of y^3 equal to 0, finding the value of z. This value of z substituted in the given equation will give the form

$$y^4 + my^2 + ny + r = 0.$$

2. Assume $y^4 + my^2 + ny + r = (y^2 + cy + f)(y^2 + ey + g)$, and deter-

* It is better for the student to use Cardan's *process* than to substitute in the formula. Thus for $x^3 - 9x + 28 = 0$, we have, by putting $x = y + z$, $y^3 + z^3 + (3yz - 9)(z + y) + 28 = 0$; and making $3yz - 9 = 0$, or $z = \frac{3}{y}$, $y^3 + \frac{27}{y^3} + 28 = 0$. Whence $y = -1$, and -3, and $z = \frac{3}{y} = -3$, and -1. $\therefore x = y + z = -4$. Then $(x^3 - 9x + 28) + (x + 4) = x^2 - 4x + 7 = 0$; whence $x = 2 \pm \sqrt{-3}$.

† An equation of the form $x^{3m} + ax^{2m} + bx^m + c = 0$ can be reduced to a cubic of the form $y^3 + my + n = 0$, by putting $x^m = y - \frac{1}{3}a$.

mine the quantities c, e, f, and g, so that they will fulfill the required conditions. Thus, expanding we have

$$y^4 + my^2 + ny + r = y^4 + \begin{array}{c|c|c|l} c & y^3 + f & y^2 + cf & y + fg; \\ +e & +ce & +eg & \\ & +g & & \end{array}$$

whence, as the members are identical,

$$c + e = 0, \quad f + ce + g = m, \quad cf + eg = n, \quad \text{and} \quad fg = r.$$

From the first we see that $c = -e$. Substituting this value, we have

$$(1)\ f - e^2 + g = m; \quad (2)\ e(f - g) = n; \quad \text{and}\ (3)\ fg = r.$$

From (1) and (2) we have $g = \frac{1}{2}\left(e^2 - \frac{n}{e} + m\right)$, and $f = \frac{1}{2}\left(e^2 + \frac{n}{e} + m\right)$; which substituted in (3) give $\left(e^2 + \frac{n}{e} + m\right)\left(e^2 - \frac{n}{e} + m\right) = e^4 + 2me^2 - \frac{n^2}{e^2} + m^2 = 4r$, or

$$e^6 + 2me^4 + (m^2 - 4r)e^2 - n^2 = 0. \qquad (4)$$

Now (4) can be reduced to a cubic in terms of e_1 by putting $e^2 = e_1 - \frac{2}{3}m$ (see foot-note on preceding page). This cubic equation will have at least *one real root* (*238*), and this will give real values to e_1, and hence to e, c, f, and g. Wherefore, *if Cardan's method gives a practical solution of* (4), we can resolve the biquadratic.

283. Sch.—It will be observed that this resolution of a biquadratic involves the resolution of a cubic, and hence is subject to the difficulty attending the irreducible case of cubics. We will give a single example, to which the student can apply the process of Descartes.

Ex.—Find by Descartes's method the roots of $x^4 - 10x^2 - 20x - 16 = 0$.

Recurring Equations.

284. A Recurring Equation is an equation such that the coefficients equidistant from the first and last are numerically equal, when the equation is in the complete form $Ax^n + Bx^{n-1} + Cx^{n-2}$ - - - - $L = 0$; and the signs of the corresponding terms are either all alike, or all unlike; *i. e.*, the coefficients of the first half recur in an inverse order in the second half of the function.

Ill. $12x^5 + 3x^4 - 5x^3 - 5x^2 + 3x + 12 = 0$ is a recurring equation. $Ax^n + Bx^{n-1} + Cx^{n-2}$ - - - - $Cx^2 + Bx + A = 0$ is the type of such equations.

285. Prop. 1.—*The roots of a recurring equation are reciprocals of each other;* i. e., *if* a *is a root,* $\frac{1}{a}$ *is also, and so of each of the roots.*

DEM.—If a satisfies the equation

$$Ax^n + Bx^{n-1} + Cx^{n-2} \cdots Cx^2 + Bx + A = 0,$$

$\frac{1}{a}$ will also satisfy it, for the former when substituted gives

$$Aa^n + Ba^{n-1} + Ca^{n-2} \cdots Ca^2 + Ba + A = 0;$$

and the latter gives

$$\frac{A}{a^n} + \frac{B}{a^{n-1}} + \frac{C}{a^{n-2}} \cdots \frac{C}{a^2} + \frac{B}{a} + A = 0,$$

which, by multiplying by a^n becomes

$$A + Ba + Ca^2 \cdots Ca^{n-2} + Ba^{n-1} + Aa^n = 0,$$

a result identical with that obtained when a is substituted.

286. SCH.—From this relation among the roots of recurring equations, they are often called *Reciprocal Equations.*

287. COR. 1.—*If the degree of the equation is* ODD *the corresponding coefficients may all have like, or all unlike, signs; but, if the degree is* EVEN *they must have like signs unless the middle term is wanting, in which case they may have unlike signs, and the roots still be reciprocal.*

That the signs may be unlike in the cases specified is evident since, if in such cases a is a root, and we substitute $\frac{1}{a}$ instead of a, clear of fractions, and change all the signs, we shall have the same result as if a had been substituted. Thus, if substituting a gives $Aa^5 + Ba^4 - Ca^3 + Ca^2 - Ba - A = 0$, substituting $\frac{1}{a}$ will give $\frac{A}{a^5} + \frac{B}{a^4} - \frac{C}{a^3} + \frac{C}{a^2} - \frac{B}{a} - A = 0$; whence clearing of fractions and changing all the signs we have $-A - Ba + Ca^2 - Ca^3 + Ba^4 + Aa^5 = 0$, a result identical with the former. The fact concerning the equation of an even degree is shown in a similar manner. Notice that *all* the corresponding coefficients must have like signs or *all* unlike signs.

288. COR. 2.—*A recurring equation may always be reduced to a form having the coefficient of the highest power of the unknown quantity, and the absolute term each* 1, *since by definition these are numerically equal.*

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289. Prop. 2.—*A recurring equation of an odd degree has one of its roots* -1 *if the signs of the corresponding terms are alike, and* $+1$ *if they are unlike.*

DEM.—Having $x^n \pm Ax^{n-1} \pm Bx^{n-2} \pm Cx^{n-3} \cdots \pm Cx^3 \pm Bx^2 \pm Ax \pm 1 = 0$,*

* The sign of x^n can always be made +. The ambiguous signs are to be taken + or −, according to the hypothesis.

taking the signs of the corresponding terms alike we can write

$$(x^n + 1) \pm Ax(x^{n-2} + 1) \pm Bx^2(x^{n-4} + 1) \pm Cx^3(x^{n-6} + 1) + \text{etc.} = 0,$$

which is divisible by $x + 1$ (PART I., ***119***), wherefore -1 is a root (***231***).

Taking the signs of the corresponding terms unlike, we can write

$$(x^n - 1) \pm Ax(x^{n-2} - 1) \pm Bx^2(x^{n-4} - 1) \pm Cx^3(x^{n-6} - 1) + \text{etc.} = 0,$$

which is divisible by $x - 1$ (PART I., ***119***), wherefore $+1$ is a root (***231***).

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290. Prop. 3.—*A recurring equation of an even degree, whose corresponding terms have opposite signs, has one root* $+1$, *and one root* -1.

DEM.—Having $x^{2n} \pm Ax^{2n-1} \pm Bx^{2n-2} \pm Cx^{2n-3} \;.\;.\;.\;.\; \mp Cx^3 \mp Bx^2 \mp Ax - 1 = 0$, taking the signs of the corresponding terms unlike, and remembering that the middle term, which would have no corresponding term, is wanting (***287***), we can write

$$(x^{2n} - 1) \pm Ax(x^{2n-2} - 1) \pm Bx^2(x^{2n-4} - 1) \pm Cx^3(x^{2n-6} - 1) + \text{etc.} = 0,$$

which is divisible by $x^2 - 1$ (PART I., ***119***); wherefore $x^2 - 1 = 0$, and $x = +1$ and -1.

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291. Prop. 4.—*A recurring equation of an even degree above the second, may be reduced to an equation of half that degree, when the signs of the corresponding terms are alike.*

DEM.—Having $x^{2n} \pm Ax^{2n-1} \pm Bx^{2n-2} \pm Cx^{2n-3} \;.\;.\;.\;.\; \pm Mx^n \;.\;.\;.\; \pm Cx^3 \pm Bx^2 \pm Ax + 1 = 0$, taking the signs of the corresponding terms alike, we can write

$$(x^{2n} + 1) \pm A(x^{2n-1} + x) \pm B(x^{2n-2} + x^2) \pm C(x^{2n-3} + x^3) + \text{etc.} = 0;$$

whence, dividing by x^n, we have

$$\left(x^n + \frac{1}{x^n}\right) \pm A\left(x^{n-1} + \frac{1}{x^{n-1}}\right) \pm B\left(x^{n-2} + \frac{1}{x^{n-2}}\right) \pm C\left(x^{n-3} + \frac{1}{x^{n-3}}\right)$$

$$\;.\;.\;.\;.\; L\left(x + \frac{1}{x}\right) \pm M = 0.$$

Now putting $x + \frac{1}{x} = y$, we can write $\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2} = y^2$, whence $x^2 + \frac{1}{x^2} = y^2 - 2$. $\left(x + \frac{1}{x}\right)^3 = x^3 + 3x^2\frac{1}{x} + 3x\frac{1}{x^2} + \frac{1}{x^3} = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = y^3$, whence $x^3 + \frac{1}{x^3} = y^3 - 3y$.

$\left(x^2 + \frac{1}{x^2}\right)^2 = x^4 + 2 + \frac{1}{x^4} = (y^2 - 2)^2$, whence $x^4 + \frac{1}{x^4} = (y^2 - 2)^2 - 2$.

$\left(x + \frac{1}{x}\right)^5 = x^5 + 5x^4\frac{1}{x} + 10x^3\frac{1}{x^2} + 10x^2\frac{1}{x^3} + 5x\frac{1}{x^4} + \frac{1}{x^5} = x^5 + \frac{1}{x^5} + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right) = y^5$, whence $x^5 + \frac{1}{x^5} = y^5 - 5(y^3 - 3y) - 10y$.

$$\left(x^3 + \frac{1}{x^3}\right)^2 = x^6 + 2 + \frac{1}{x^6} = (y^3 - 3y)^2, \text{ whence } x^6 + \frac{1}{x^6} = (y^3 - 3y)^2 - 2.$$

Whence we see that any term of the form $x^n + \frac{1}{x^n}$ may be expressed in terms of y, and will involve no higher power than y^n. Therefore the original equation, which is of the $2n$th degree, can by this substitution be transformed into an equation in y, of the nth degree.

EXAMPLES.

Solve the following recurring equations by applying the foregoing principles:

1. $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0.$
2. $x^5 - 11x^4 + 17x^3 + 17x^2 - 11x + 1 = 0.$
3. $6x^5 - 11x^4 - 33x^3 + 33x^2 + 11x - 6 = 0.$
4. $1 + x^5 = a(1 + x)^5.$
5. $x^7 - 2x^5 + x^4 + x^3 - 2x^2 + 1 = 0.$
6. $8x^6 - 16x^4 - 25x^3 - 16x^2 + 8 = 0.$
7. $4x^6 - 24x^5 + 57x^4 - 73x^3 + 57x^2 - 24x + 4 = 0.$
8. $x^4 + 4ax^3 - 19a^2x^2 + 4a^3x + a^4 = 0.$
9. $x^4 + x^3 + x^2 + x + 1 = 0.$
10. $1 + x^4 = \frac{1}{2}(1 + x)^4.$

BINOMIAL EQUATIONS AND THE ROOTS OF UNITY.

292. A Binomial Equation is one of the form $x^n \pm a = 0$. Such equations may be considered as recurring equations and solved accordingly.

ILL.—Having $x^n \pm a = 0$, put $x^n = ay^n$; whence $ay^n \pm a = 0$, or $y^n \pm 1 = 0$, which is recurring.

EXAMPLES.

1. $x^3 \pm 5 = 0.$
2. $x^4 \pm 3 = 0.$
3. $x^5 \pm 2 = 0$
4. $x^6 \pm 7 = 0.$
5. $x^8 \pm 11 = 0.$
6. $x^9 \pm 1 = 0.$

7. What are the *two* square roots of 1? The *three* cube roots of 1? The *four* fourth-roots of 1? The *five* fifth-roots of 1? The *six* sixth-roots of 1?

SUG.—The solution of these questions consists in resolving $x^2 - 1 = 0$, $x^3 - 1 = 0$, $x^4 - 1 = 0$, etc. The five fifth-roots of 1 are

$$1, \ \tfrac{1}{4}\left(\sqrt{5} - 1 \pm \sqrt{-10 - 2\sqrt{5}}\right), \text{ and } -\tfrac{1}{4}\left(\sqrt{5} + 1 \pm \sqrt{-10 + 2\sqrt{5}}\right).$$

293. SCH.—It will be observed that the form $x^7 \pm 1 = 0$ is omitted above. Now $x^7 - 1 = 0$ has one root 1. The equation can therefore be depressed to a recurring equation of the 6th degree, having all its signs +. This can be reduced to a cubic by (*291*). $x^7+1=0$ has one root $x=-1$, and can be reduced to a recurring equation of the 6th degree having its signs alternately + and −. This can be resolved into one of the 3'rd degree by (*291*). Hence the complete resolution of $x^7 \pm 1 = 0$ depends on the resolution of a cubic.

$x^9 \pm a = 0$ can be resolved by putting $x^3 = y$, whence we have $y^3 \pm a = 0$. Solving this for y we have 3 roots. Call them a_1, a_2, a_3. Hence to complete the solution we have to resolve the three cubics $x^3 \pm a_1 = 0$, $x^3 \pm a_2 = 0$, $x^3 \pm a_3 = 0$.

EXPONENTIAL EQUATIONS.

294. *Exponential Equations* are equations in which the unknown quantity or quantities are involved in the exponents.

ILL. $a^x + b^y = c$, $a^x = d$, $2^x = 42$, $3^{\frac{1}{y}} = 2$, $y^x = 256$, $x^x = 100$, and $x^y - y^x = m$ are exponential equations.

295. *Prob. 1.*—*To solve an exponential equation of the form* $a^x = m$.

SOLUTION.—Taking the logarithms of both members we have $x \log a = \log m$ (*180, 181*); whence $x = \dfrac{\log m}{\log a}$. Therefore finding the logarithms of m and a from a table of logarithms, and dividing the former by the latter, we find x.

296. *Prob. 2.*—*To solve an exponential equation of the form* $x^x = m$.

SOLUTION.—Taking the logarithms of both members we have $x \log x = \log m$. Then find $\log m$ from the table, and determine x by inspection from the table so that $x \times \log x$ shall equal $\log m$ exactly or approximately.*

EXAMPLES.

1. Find the value of x in the equation $3^x = 2546$.

SOLUTION. $x \log 3 = \log 2546$. $\therefore x = \dfrac{\log 2546}{\log 3} = \dfrac{3.405858}{.477121} = 7.138 +$.

2 to 6. Solve the following: $(24)^y = 18742$; $2^x = 2673$; $(11)^y = 2681$; $2^x = 10$; $5^x = 1$; $(12)^y = 1$.

7. Find the value of x in the equation $x^x = 3561$.

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* The method of solving such equations by Double Position is entirely useless, since a table of logarithms is necessary for that method, and having such a table at hand, the approximations can be made to any extent likely to be desired, more readily by simple inspection than by computing the errors by Double Position. Moreover, the method here given affords an excellent exercise in the use of the tables.

SOLUTION.—We have $x \log x = \log 3561 = 3.551572$. Now looking in a table of logarithms, we soon see that x must be near 5, since $5 \log 5 = 5 \times .698970 = 3.494850$. Thus we see that $x > 5$. Trying 5.1 we have $5.1 \log 5.1 = 3.608607$, $\therefore x < 5.1$. Therefore we try 5.05. $5.05 \log 5.05 = 3.55161955$, which coincides so nearly with the required value of $x \log x$, that undoubtedly the 100ths figure is 4. Again, for a nearer approximation try 5.049, as the value of x is very near 5.05. $5.049 \log 5.049 = 3.550482$. Hence we see that $x = 5.049 +$.

8 to 15. Solve the following as above: $x^x = 100$; $x^x = 7$; $x^x = 21$; $x^{2x} - 40x^x = 200$; $3^{2x} + 3^x = 100$; $a^x - \dfrac{c}{a^x} = 2b$; $a^{bx-2} = c$; $a^{mx}b^{nx} = c$.

16 to 21. Solve the following: $x^y = y^x$, and $x^3 = y^2$; $x^y = y^x$, and $x^a = y^b$; $m^{x-y} = n$, and $x + y = q$; $2^y 3^x = 500$, and $2x = 3y$; $5^{x^2-x} = 256$; $(a^4 - 2a^2b^2 + b^4)^{x-1} = (a-b)^{2x}(a+b)^{-2}$.

22. Given the fundamental formulæ of Geometrical Progression, viz., $l = ar^{n-1}$, and $S = \dfrac{lr - a}{r - 1}$, to find the following:

$$n = \frac{\log l - \log a}{\log r} + 1; \qquad n = \frac{\log [a + (r-1)S] - \log a}{\log r};$$

$$n = \frac{\log l - \log a}{\log (S-a) - \log (S-l)} + 1, \text{ and } n = \frac{\log l - \log [lr - (r-1)S]}{\log r} + 1.$$

23. Given the two fundamental formulæ of Compound Interest, viz., $a = p(1+r)^t$,* and $i = a - p$, to find the following:

$$t = \frac{\log (p+i) - \log p}{\log (1+r)}; \qquad t = \frac{\log a - \log p}{\log (1+r)}; \qquad \log (1+r) = \frac{\log (p+i) - \log p}{t};$$

$$\log (1+r) = \frac{\log a - \log p}{t};$$

$$t = \frac{\log a - \log (a-i)}{\log (1+r)}; \qquad \log (1+r) = \frac{\log a - \log (a-i)}{t}.$$

NOTE.—Many problems in Compound Interest, Annuities, and kindred subjects are most expeditiously solved by means of logarithms. The student who has not a table of logarithms at hand may either omit the following examples in this section, or content himself with selecting the proper formula and telling how it is applied to the solution of the particular example.

24. What is the amount of \$100 at 7% annual compound interest

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* This formula is obtained thus: letting r represent the rate for time 1, expressed decimally, *i. e.*, if the rate is 7 per ct., $r = .07$, or $\frac{7}{100}$, we have for time 1 (as 1 year), $a = p + pr = p(1+r)$; for time 2, $a = p(1+r) + pr(1+r) = p(1+r)^2$; for time 3, $a = p(1+r)^2 + pr(1+r)^2 = p(1+r)^3$; therefore for time t, $a = p(1+r)^t$.

for 10 years? What if the interest is compounded semi-annually? What if quarterly? What in each case if the rate is 10%? If 6%? If 3%?

SUG'S.—We have $a = p(1+r)^t$, whence $\log a = \log p + t \log(1+r) = \log 100 + 20 \log 1.035$, for interest at 7% compounded semi-annually.

25. In what time will a sum of money double itself at 10% compounded semi-annually? At 7% compounded annually? In what time triple? Quadruple?

SUG. $a = 2p = p(1+r)^t$, whence $2 = (1+r)^t$, and $t = \dfrac{\log 2}{\log(1+r)}$.

26. In what time will \$10 amount to \$100 at 8% compounded annually?

27. What is the present worth of \$2000 due 3 years hence, without interest, if money is worth 10% compound interest?

SUG.—The present worth is a sum which, put at compound interest at 10%, will amount to \$2000 in 3 years. Hence $2000 = p(1.1)^3$, p standing for present worth. Whence $\log p = \log 2000 - 3 \log(1.1)$.

28. A soldier's pension of \$350 per annum is 5 years in arrears. Allowing 5% compound interest, what is now due him?

SUG'S.—The 5th, or last year's unpaid pension has no interest on it, as it is just due. The 4th, or next to the last, has 1 year's interest due, and hence amounts to 350 (1.05). The 3d year's pension has 2 years' interest due, and hence amounts to $350(1.05)^2$. Thus the total is found to be $350 + 350(1.05) + 350(1.05)^2 + 350(1.05)^3 + 350(1.05)^4$, or $350\left\{1 + (1.05) + (1.05)^2 + (1.05)^3 + (1.05)^4\right\}$

29. Letting S represent the amount of an annuity a, in arrears for t years, compound interest being allowed, at r%, show that $S = a \cdot \dfrac{(1+r)^t - 1}{r}$.

30. What is the present worth of an annuity of \$200 for 7 years, money being worth 5% compound interest?

SUG.—Evidently, a sum which, put at 5% compound interest, will amount to the same sum in 7 years, as the annuity will.

31. Letting P be the present worth of an annuity a, for time t, at r% compound interest, show that $P = \dfrac{a}{r} \cdot \dfrac{(1+r)^t - 1}{(1+r)^t}$. Also, that if the annuity is perpetual (runs forever), $P = \dfrac{a}{r}$.

SUG.—When $t=\infty$, $P=\frac{a}{r}\cdot\frac{(1+r)^{\infty}-1}{(1+r)^{\infty}}=\frac{a}{r}\cdot\frac{(1+r)^{\infty}}{(1+r)^{\infty}}=\frac{a}{r}$, as it evidently should, since such an annuity is worth a present sum which will yield an annual interest equal to the annuity.

32. What is the present worth of a perpetual annuity of \$350, money being worth $7\frac{3}{10}$% compound interest? If money is worth 10% compound interest?

33. What is the present worth of an annual pension of \$125, which commences 3 years hence* (first *payment* to be made 4 years hence), and runs 10 years, money being worth 10% compound interest?

SUG.—Evidently, the difference between the present worth of such a pension for 13 years, and for 3 years.

34. An annuity a, which commences T years hence, and runs t years at r% compound interest, gives

$$P=\frac{a}{r}\left\{\frac{(1+r)^{T+t}-1}{(1+r)^{T+t}}-\frac{(1+r)^{T}-1}{(1+r)^{T}}\right\}=\frac{a}{r}\left\{(1+r)^{-T}-(1+r)^{-(T+t)}\right\}.$$

When the annuity is perpetual after the time T, we have $P=\frac{a}{r}(1+r)^{-T}$. Student give proof.

35. Two sons are left, one with the immediate possession of an estate worth \$12000, and the other with a perpetual annuity of \$800 in reversion after 7 years: money being worth 5% compound interest, which has the more valuable inheritance, and how much?

36. What annual payment will meet principal and interest of a debt of \$2000 at 8% compound interest in 5 years?

SUG'S.—The amount of \$2000 at 8% compound interest for 5 years = the amount of the annuity a for the same rate and time.

37. Show that if D is a debt at compound interest at r%, b an annual payment, and t the number of years required to liquidate the debt, $t=\frac{\log b-\log(b-Dr)}{\log(1+r)}$.

38. The debt of a certain State is \$20,000,000, bearing annual interest at $4\frac{1}{2}$%. A sinking fund of \$2,000,000 annually is set apart to meet it. How long will it require to extinguish the debt? How long if instead of paying the \$2,000,000 annually on the debt, it is invested at 6% compound interest?

* An annuity which commences after some specified time is said to be *in reversion.*

39. A farmer has paid \$10 per annum for newspapers, which he considers have increased his net annual income at least $\frac{1}{10}$. For 10 years during which his net income has been \$500 annually, money has been worth 10% compound interest. What is the total net gain to be credited to his investment in newspapers?

40. A boy commenced smoking when 15 years old. For the first 5 years he smoked 2 5-cent cigars each day. For the next 20 years, 3 10-cent cigars per day. Now had he abstained from smoking and invested at the end of each six months the amount thus saved, at 10% annual compound interest, how much would he have accumulated from this source at the age of 40?

41. A man pays a premium of \$104 per annum on a life policy of \$4200 for 20 years before his death. Money being worth 10% compound interest, does the insurance company gain or lose, and how much?

CHAPTER IV.

DISCUSSION, OR INTERPRETATION, OF EQUATIONS.

297. ***To Discuss, or Interpret, an Equation or an Algebraic Expression,*** is to determine its significance for the various values, absolute or relative, which may be attributed to the quantities entering into it, with special reference to noting any changes of values which give changes in the general significance.

Such discussions may be divided into two classes: 1st. The discussion of equations or expressions with reference to their constants; and 2d. The discussion of equations or expressions with reference to their variables.

The following principles are of constant use in such discussions:*

298. ***Prop.***—*A fraction, when compared with a finite quantity, becomes:*

* These principles, and in fact most of this chapter, have been considered previously, but are collected here for review and connected study.

1. *Equal to* 0, *when its numerator is* 0 *and its denominator finite, and when its numerator is finite and its denominator* ∞.

2. *Equal to* ∞, *when its numerator is finite and its denominator* 0, *and when its numerator is* ∞ *and its denominator finite.*

3. *It assumes an indeterminate form when numerator and denominator are both* 0, *and when they are both* ∞.*

DEM.—These facts appear when we consider that the value of a fraction depends upon the relative magnitudes of numerator and denominator.

1. Let a be any constant and x a variable, then the fraction $\frac{x}{a}$ diminishes as x diminishes, and becomes 0 when x is 0. Again, the fraction $\frac{a}{x}$ diminishes as x increases, and when x becomes ∞, *i. e.*, greater than any assignable magnitude, $\frac{a}{x}$ becomes less than any assignable magnitude or infinitesimal, and is to be regarded as 0 in comparison with finite quantities. (See ***142*** and ***151***, DEM., and foot-note.)

2. As x increases, the fraction $\frac{x}{a}$ increases, and hence when x becomes infinite the value of the fraction is infinite. Also as x diminishes the value of $\frac{a}{x}$ increases; hence when x becomes infinitely small, or 0, the value of the fraction exceeds any assignable limits, and is therefore ∞.

3. Finally, if x and y are variables, $\frac{x}{y}$ *diminishes* as x diminishes, and *increases* as y diminishes. What then does it become when $x = 0$, and $y = 0$? *i. e.*, what is the value of $\frac{0}{0}$? Simple arithmetic would lead us to suppose that $\frac{0}{0}$ was *absolutely* indeterminate, *i. e.*, that it might have any value whatever assigned to it, for $\frac{0}{0} = 5$, since $0 = 5 \times 0 = 0$; $\frac{0}{0} = 7$, since $0 = 7 \times 0 = 0$, etc. But a closer inspection will enable us to see that the symbol $\frac{0}{0}$ is not necessarily indeterminate, or rather that the expression which takes this form for particular values of its components, has not necessarily an indefinite number of values for these values of its components. Thus, what the value of $\frac{x}{y}$ will be when x and y each diminish to 0 will evidently depend upon the relative values of x and y at first, and which diminishes the faster. Suppose, for example, that $y = 5x$; then $\frac{x}{y} = \frac{x}{5x}$. Now, suppose x to diminish; the denominator will diminish 5

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* By this is meant that $\frac{0}{0}$ and $\frac{\infty}{\infty}$ *may* have a variety of values, not that they necessarily *do* have.

times as fast as the numerator, and whatever the value of x, the value of the fraction will be $\frac{1}{7}$. So if $y = 7x$, $\frac{x}{y} = \frac{x}{7x}$, which is $\frac{1}{7}$ for any value of x. Hence when $x = 0$, and $y = 0$, we have $\frac{x}{y} = \frac{0}{0} = \frac{x}{5x} = \frac{1}{5}$, or $\frac{x}{y} = \frac{0}{0} = \frac{x}{7x} = \frac{1}{7}$, or $\frac{x}{y} = \frac{0}{0}$ = any other value depending upon the relative values of x and y. So, also, if $x = \infty$, and $y = \infty$, $\frac{x}{y} = \frac{\infty}{\infty}$; but if $y = 6x$, we have $\frac{x}{y} = \frac{\infty}{\infty} = \frac{x}{6x} = \frac{1}{6}$. And so if $y = 10x$, we have $\frac{x}{y} = \frac{\infty}{\infty} = \frac{x}{10x} = \frac{1}{10}$. Thus we see that the *mere* fact that numerator and denominator become 0, or become ∞, does not determine the value of the fraction, *i. e.*, gives it an indeterminate form.

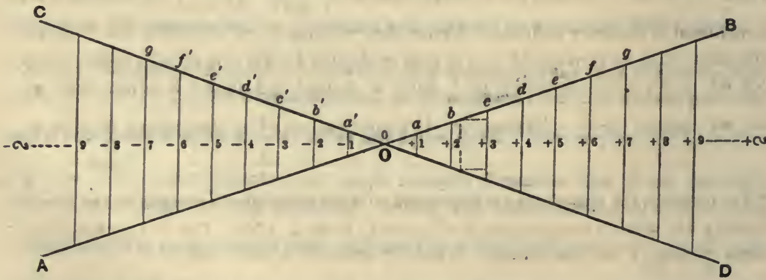
299. *A Real Number or Quantity* is one which may be conceived as lying somewhere in the series of numbers or quantities between $-\infty$ and $+\infty$ inclusive.

ILL.—Thus, if we conceive a series of numbers varying both ways from 0, *i. e.* positively and negatively to ∞, we have

$$-\infty \text{ - - - - } -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4, \text{ - - - - } +\infty.$$

Now a *real number* is one which may be conceived as situated somewhere within these limits; it may be +, −, integral, fractional, commensurable, or incommensurable. Thus + 15624 and − 15624 will evidently be found in this series. $+\frac{17}{3}$ may be conceived as somewhere between + 5 and + 6, though *its exact locality* could not be fixed by the arithmetical conception of discontinuous number. So, also, $-\frac{17}{3}$ is somewhere between − 5 and − 6. Again $+\sqrt{5}$ is somewhere between + 2 and + 3, though, as above, we cannot locate it exactly by the arithmetical conception.

The following *Geometrical Illustration* is more complete than the arithmetical. Thus let two indefinite lines, as CD and AB, intersect (cross) each other, as at O, Now let parallel, equidistant lines be drawn between them. Call the one at a



+1, that at b will be +2, at c +3, etc. So, also, the line at a' being −1, that at b' will be −2, at c' −3, etc. Now conceive one of these lines to start from an infinite distance at the left and move toward the right. When at an infinite distance to the left of O its value would be $-\infty$, and in passing to O it would pass through *all possible negative values.* In passing O it becomes 0 at O, changes sign to + as it passes, and moving on to infinity to the right, passes through *all possible positive values.* Hence we see how *all* real values are embraced between $-\infty$ and $+\infty$ inclusive.*

300. An Imaginary Number or Quantity is one which cannot be conceived as lying anywhere between the limits of $-\infty$ and $+\infty$, as explained above. The *algebraic form* of such a quantity is an expression involving an even root of a negative quantity.† (See PART I., ***218.***)

EXAMPLES.

1. What are the values of x and y in the expressions $x = \frac{b' - b}{a - a'}$, $y = \frac{ab' - a'b}{a - a'}$, when $b = b'$ and a and a' are unequal? When $b = b'$ and $a = a'$? When $a = a'$ and b and b' are unequal? What are the *signs* of x and y when $b > b'$ and $a > a'$, the essential signs of a, a', b, and b' being +? When $b > b'$ and $a < a'$? If a' and b are essentially negative, and $a = a'$, and $b = b'$, what are the values of x and y? If a' and b' are each 0?

2. What *general* relation between a and a' renders $\frac{a' - a}{1 + aa'} = 0$? What renders it ∞?

SOLUTION.—To render $\frac{a' - a}{1 + aa'} = 0$, we must have $a' - a = 0$, and $1 + aa'$ finite or infinite; or else we must have $1 + aa' = \infty$, while $a' - a$ is finite or 0 (***298***). Now $a' - a = 0$ gives $a' = a$; whence $\frac{a' - a}{1 + aa'} = \frac{0}{1 + a^2}$, which is 0 for any value of a finite or infinite. Hence the relation $a' = a$ fulfills the first requirement. Let us now see if $1 + aa' = \infty$ will also fulfill this requirement. This gives $aa' = \infty$, since subtracting 1 from ∞ would not make it other than ∞. Thus we have $a' = \frac{\infty}{a}$. Hence for all finite values of a (including 0) a' is ∞,

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* For example, the student who is acquainted with the elements of geometry knows how to construct a line which is exactly equal to $\sqrt{5}$ (GEOM., PART I., ***110***). This line he can locate between +2 and +3, and also between −2 and −3, since $\sqrt{5}$ is both + and −.

† Transcendental functions afford other forms of imaginary expressions; for example, $\sin^{-1} 2$, $\sec^{-1} \frac{1}{2}$, $\log(-120)$, $\log(-m)$, etc. But our limits forbid the consideration of the interpretation of imaginaries, except in the most restricted sense, as indicating incompatibility with the arithmetical sense of the problem.

and $\frac{a'-a}{1+aa'}=\frac{a'}{aa'}$* $=\frac{1}{a}$, which can only be 0 when $a=\infty$. Therefore the *particular* values $a'=\infty=a=\infty$, render $\frac{a'-a}{1+aa'}=0$; but no *general* values do.

Again, in order that $\frac{a'-a}{1+aa'}=\infty$, we must have $1+aa'=0$, and $a'-a$ finite or infinite; or else we must have $a'-a=\infty$, and $1+aa'$ finite or 0. Now $1+aa'=0$ gives $a=-\frac{1}{a'}$; $\frac{a'-a}{1+aa'}=\frac{a'+\frac{1}{a'}}{1-\frac{a'}{a'}}=\frac{a'^2+1}{a'-a'}=\frac{a'^2+1}{0}=\infty$ for any value of a' finite or infinite. Therefore the general relation $a=-\frac{1}{a'}$ between a and a' renders $\frac{a'-a}{1+aa'}=\infty$.† Let us now see if the relation $a'-a=\infty$ will do the same. Now if $a'-a=\infty$, one or the other (a' or a) must be ∞. Let $a'=\infty$. We then have $\frac{a'-a}{1+aa'}=\frac{a'}{aa'}=\frac{1}{a}$, which can only be ∞ when $a=0$. Hence the *particular* values $a'=\infty$ and $a=0$ render $\frac{a'-a}{1+aa'}=\infty$, but no general values meet the requirement unless $a=-\frac{1}{a'}$.

3. What general relation between a and a' renders $\frac{1-aa'}{a'+a}=0$? What renders it ∞?

4. In the expression $y=-2x+4\pm\sqrt{x^2-4x-5}$, how many values has y, *in general*, for any particular value of x? For what value or values of x has y but one value? For what values of x is y real? For what imaginary? For what values of x is y positive? For what negative?

SOLUTION.—Writing the expression thus, $y=-(2x-4)\pm\sqrt{x^2-4x-5}$, we see that the value of y is made up of two parts, viz., a rational part $-(2x-4)$, and a radical part $\sqrt{x^2-4x-5}$. But the radical part may be taken with either the + or the − sign. Hence, *in general*, for any particular value of x *there are two values of y*. 2d. But if such a value is given to x as to render the radical part 0, for this value of x, y will have but one value, viz., the rational part. But the condition $\sqrt{x^2-4x-5}=0$ gives $x=5$ and -1. Thus for

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* This reduction is made by dropping a and 1, since the subtraction of a finite from an infinite, or the addition of a finite to an infinite, does not change the character of the infinite. Thus, in this case, to assume that dropping a and 1 affected the relation between numerator and denominator, would be to assign to a and 1 some values with respect to the infinite a'. But this is contrary to the definition of an infinite.

† It is to be observed that the relation $a=-\frac{1}{a'}$ requires that a and a' have different essential signs; while the relation $a'=a$ requires that they have the same essential signs.

$x = 5$, $y = -6$, but *one* value; and for $x = -1$, $y = +6$, also but *one* value. 3d. To ascertain for what values of x, y is *real*, we observe that y is real when $x^2 - 4x - 5$ is positive, and imaginary when $x^2 - 4x - 5$ is negative. Now for x positive $x^2 - (4x + 5)$ is + when $x^2 > 4x + 5$; and for x negative, we have $x^2 + 4x - 5$, which is positive when $x^2 + 4x > 5$. The former inequality gives $x^2 - 4x + 4 > 9$, or $x > 5$; and the latter gives $x^2 + 4x + 4 > 9$, or $x > 1$. Hence for positive values of x greater than 5, y is real, and for negative values of x numerically greater than 1, y is real. The 4th inquiry is answered by this: y is imaginary for all values of x between -1 and $+5$. 5th. To ascertain what + values of x render y +, and what −, we observe that $-(2x-4) \pm \sqrt{x^2-4x-5}$ can only be + when the + sign of the radical part is taken and when $\sqrt{x^2 - 4x - 5} > 2x - 4$. This gives $x < 2 \pm \sqrt{-3}$, *i. e.*, an imaginary quantity. Hence y is never + for x +. Taking the negative sign of the radical we see that both parts of the value of y are −, and consequently y is real and negative for all + values of x which render y real, *i. e.*, for values greater than 5. Finally, for x − we have $y = 2x + 4 \pm \sqrt{x^2 + 4x - 5}$. Now when we take the + sign of the radical both parts are +; hence this value of y is always +. When we take the − sign of the radical y is negative if $2x + 4 < \sqrt{x^2 + 4x - 5}$. But this gives $x < -2 \pm \sqrt{-3}$. Hence y is never negative for any negative value of x. Therefore both values of y are positive and real for all negative values of x numerically greater than 1.

5 to 22. Discuss as above the values of y in the following; *i. e.*, 1st. Show how many values y has *in general*, and whether they are equal or unequal; 2d. For what particular value or values of x, y has but one value; 3d. For what values of x, y is real, and for what imaginary; 4th. For what values of x, y is +, and for what −; 5th. Also determine what values of x render y infinite:

(5.) $y^2 + 2xy - 2x^2 - 4y - x + 10 = 0$;*

(6.) $y^2 - 2xy + 2x^2 - 2y + 2x = 0$;

(7.) $y^2 + 2xy + x^2 - 6y + 9 = 0$;

(8.) $y^2 + 2xy + 3x^2 - 4x = 0$;

(9.) $y^2 - 2xy + 3x^2 + 2y - 4x - 3 = 0$;

(10.) $y^2 + 2xy - 3x^2 - 4x = 0$;

(11.) $y^2 - 2xy + x^2 + x = 0$;

(12.) $y^2 - 2xy + x^2 - 4y + x + 4 = 0$;

(13.) $y^2 - 2xy + x^2 + 2y + 1 = 0$;

(14.) $y^2 - 2x^2 - 2y + 6x - 3 = 0$;

(15.) $y^2 - 2xy - 3x^2 - 2y + 7x - 1 = 0$;

(16.) $y^2 - 2xy - 2 = 0$;

(17.) $y^2 - 2xy + 2y + 4x - 8 = 0$;

* In all cases solve the equation for y in the first place. In this example

$$y = -x + 2 \pm \sqrt{3x^2 - 3x - 6}.$$

(18.) $4y^2 + 4x^2 + 2y - 3x + 12 = 0$;
(19.) $3y^2 - 8x^2 = 12$;
(20.) $12y^2 + 4x^2 = 20$;
(21.) $x^2 + y^2 = 16$;
(22.) $x^2 - y^2 = 20$.

23. Discuss the equation $ay^2 - x^3 + (b - c)x^2 + bcx = 0$, as above, when $b > c$; also when $c > b$.

SUG'S. $y = \pm \frac{1}{a^{\frac{1}{2}}}\sqrt{x^3 - (b-c)x^2 - bcx}$. Whence we see that y has two values for every value of x, numerically equal, but with opposite signs. y is 0, when $x^3 - (b-c)x^2 - bcx = 0$; *i. e.*, when $x = 0$, $x = b$, and $-c$. Again y is real for $x+$, when $x^3 > (b-c)x^2 + bcx$, or $x^2 > (b-c)x + bc$; which gives $x > b$. For $x-$, we have $y = \pm \frac{1}{a^{\frac{1}{2}}}\sqrt{-x^3 - (b-c)x^2 + bcx}$, which gives y real when $x^3 + (b-c)x^2 < bcx$, which gives x numerically less than c, *i. e.*, greater than $-c$. Hence y is imaginary for all values of x between 0 and $+b$, and real for all values of x from $+b$ to $+\infty$. So also y is real for all values of x from 0 to $-c$, and imaginary for all values of x from $-c$ to $-\infty$.

24. Discuss as above $y^2 = (x-a)^2\frac{x-b}{x}$, showing that in general y has two values numerically equal but with opposite signs; that it is 0 for $x = a$, and $x = b$; is imaginary from $x = 0$ to $x = b$ (except when $x = a$, b being greater than a); real from $x = b$ to $x = +\infty$, and real for *all* negative values of x, *i. e.*, from $x = 0$ to $x = -\infty$; and that for $x = 0$, $y = \pm\infty$, and for $x = +\infty$, $y = \pm\infty$; also for $x = -\infty$, $y = \pm\infty$.

25. Show from the equation $y + x^2y = x$, that $y = 0$ when $x = 0$, $+\infty$, and $-\infty$; also that y has but one value for any particular value of x; that it is $+$ when x is $+$, and $-$ when x is $-$; and that y increases numerically as x passes from 0 to $+1$, and from 0 to -1, but that it diminishes numerically as x passes from $+1$ to $+\infty$, and also from -1 to $-\infty$.

26. Discuss $y^2x = 4a^2(2a - x)$ with reference to y as a function of x, as above.

27. Show that in the equation $y^3 - 3axy + x^3 = 0$, y has three real values between the limits $x = 0$, and $x = a\sqrt[3]{4}$, and only one real value between the limits $x = a\sqrt[3]{4}$ and $x = +\infty$, and also between the limits $x = 0$ and $x = -\infty$.

SUG.—This is done by means of Cardan's formula. (See ***280.***)

301. Arithmetical Interpretations of Negative and Imaginary Solutions.

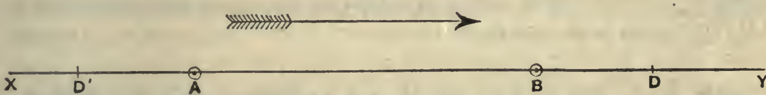
1. A is 20 years old, and B 16. When will A be twice as old as B?

Sug's.—We have $20+x=2(16+x)$; whence $x=-12$. The arithmetical interpretation of this result is that A *will never* be twice as old as B, but that he *was* twice as old 12 years ago, *i. e.*, when he was 8 and B 4.

2. A is a years old, and B, b. When will A be n times as old as B? For $n>1$ what are the possible relative values of a and b consistently with the arithmetical sense of the problem? Interpret for $a>nb$, $a=nb$, $a<nb$ when $n>1$. Also for $n=1$, $a>nb$, $a<nb$, and $a=nb$.

3. Two couriers, A and B, are traveling the same road in the same direction, the former at rate a, the latter at rate b. They are at two places c miles apart at the same time. Where and when are they together?

Solution and Discussion.—Let XY represent the road which the couriers are traveling in the direction from X to Y, and A and B the stations which they



pass at the same time, A being at A when B is at B, and D or D′ the place at which they are together. Call the distance from B to the place at which they are together $\pm x$, $+x$ when D is beyond B, and $-x$ when it is on the hither side of A and B, as at D′. Then the distance from A to the point at which they are together is $c+(\pm x)$. Now disregarding the essential sign of x, and leaving it to be determined in the sequel, we have

$$\text{Distance } A \text{ travels from A} = c+x,$$
$$\text{Distance } B \text{ travels from B} = x;$$

Time from passing A and B to the time they are together $\frac{c+x}{a}$ and $\frac{x}{b}$.

But these are equal. Hence we are to discuss the equation

$$\frac{c+x}{a}=\frac{x}{b}, \text{ or } x=\frac{bc}{a-b}, \text{ and } c+x=\frac{ac}{a-b}.$$

The points to be noticed in the discussion are, (1) when $a>b$, (2) when $a<b$, (3) when $a=b$, c being greater than 0 in each case but not ∞. Also the like cases when $c=0$.

When c >0 *but not* ∞.

We have, for $a>b$, x *positive*, which shows that the point at which they are together is at the right of B, *i. e.*, in the direction which they are traveling. The *time*, $\frac{x}{b}\left(\text{or } \frac{c+x}{a}\right)$, is *positive*, which shows that they are together *after* passing A and B.

For $a < b$, x is *negative*, and $c + x$, which equals $\frac{ac}{a-b}$, is also negative. This shows that they *were* together at a point at the left of A, that is, before they reached the stations A and B. With this the expressions for the time also agree. Thus $\frac{x}{b}$ becomes $-\frac{x}{b}$, and $\frac{c+x}{a}$ is also negative, since in this case $x > c$.

When $a = b, x = \frac{bc}{a-b} = \frac{bc}{0} = \infty$, and $c + x = \frac{ac}{a-b} = \frac{ac}{0} = \infty$; which indicates that they are never together.

When c = 0.

In this case $x = \frac{bc}{a-b} = 0$, and $c + x = \frac{ac}{a-b} = 0$, for a and b unequal, indicating that they are together when they are at A and B. This is evidently correct, since A and B coincide in this case. When $a = b, x = \frac{bc}{a-b} = \frac{0}{0}$, and $c + x = \frac{0}{0}$, which shows that they are *always* together, $\frac{0}{0}$ being a symbol of indetermination which in this instance may have any value whatever, as we see from the nature of the problem.

302. Sch.—The student should not understand that the symbol $\frac{0}{0}$ *always* indicates that the quantity which takes this form has an indefinite number of values. It is frequently so, but not necessarily. The indetermination may be only *apparent*, and what the value of the expression is must be determined from other considerations. The Calculus affords the most elegant general methods of evaluating such expressions. But the simple processes of Algebra will often suffice. Thus for $x = 1$, $\frac{1-x^3}{1-x} = \frac{0}{0}$. But $\frac{1-x^3}{1-x} = 1 + x + x^2$, which, for $x = 1$, is 3. Hence $\frac{1-x^3}{1-x} = 3$, for $x=1$. Here the apparent indetermination arises from the fact that the particular assumption (that $x = 1$) causes the two quantities between which we wish the ratio, viz., the numerator and denominator, to disappear. Let the student find that $\frac{1-x^5}{1-x+x^2-x^3} = 2\frac{1}{2}$ for $x = 1$. (See also ***298***, 3d part of demonstration.)

4. Two couriers starting at the same time from the two points A and B, c miles apart, travel *toward* each other at the rates a

and b respectively. Discuss the problem with reference to the place and time of meeting. (Consider when $a > b$, $a < b$, and $a = b$.)

5. Two couriers, A and B, are traveling the same road in the same direction, the former at rate a, and the latter n times as fast. They are at two places c miles apart at the same time. Discuss the problem with reference to place and time of meeting as in Ex. 3, adding the considerations, $n > 1$, $n < 1$, $n = 1$, $n = 0$.

6. Divide 10 into two parts whose product shall be 40.

SOLUTION AND DISCUSSION.—Let x and y be the parts, then $x + y = 10$, $xy = 40$, and $x = 5 \pm \sqrt{-15}$, $y = 5 \mp \sqrt{-15}$. These results we find to be imaginary. This signifies that the problem in its arithmetical signification is impossible: this indeed is evident on the face of it. But, although impossible in the arithmetical sense, the values thus found do satisfy the *formal*, or algebraic sense. Thus the *sum* of $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ is 10, and the product 40.

7. The sum of two numbers is required to be a, and the product b: what is the maximum value of b which will render the problem possible in the arithmetical sense? What are the parts for this value of b?

8. Divide a into two parts, such that the sum of their squares shall be a minimum.

SUG'S.—Let x and $a-x$ be the parts, and m the minimum sum. Then

$$x^2 + (a - x)^2 = 2x^2 - 2ax + a^2 = m;$$

whence $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{2m - a^2}$. From this we see that if $2m < a^2$, x is imaginary. Hence the least value which we can have is $2m = a^2$, or $m = \frac{1}{2}a^2$.

9. Divide a into two parts, such that the sum of the square roots shall be a maximum.

10. Let d be the difference between two numbers: required that the square of the greater, divided by the less, shall be a minimum.

11. Let a and b be two numbers of which a is the greater, to find a number such that if a be added to this number, and b be subtracted from it, the product of this sum and this difference, divided by the square of the number, shall be a maximum.

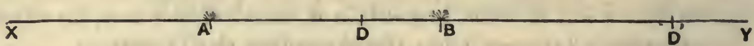
SUG'S.—Let n be the number, and m the required maximum quotient. Then by the conditions $\frac{n^2 + (a - b)n - ab}{n^2} = m$, whence we find

$$n = -\frac{a - b}{2(1 - m)} \pm \frac{\sqrt{a^2 + 2ab + b^2 - 4abm}}{2(1 - m)}.$$

From this we see that the greatest value which m can have and render n real is $m = \frac{(a+b)^2}{4ab}$. This gives $n = -\frac{a-b}{2(1-m)} = \frac{2ab}{a-b}$.

12. To find the point on a line passing through two lights at which the illumination will be the same from each light.

Solution.—Let A and B be the two lights, and XY the line passing through



them. Let a be the intensity of the light A at a unit's distance from it, b the intensity of B at a unit's distance from it, c the distance between the two lights, as AB, and x the distance of the point of equal illumination from the light A, as AD (or AD′). Then, as we learn from Physics that the illuminating effect of a light varies inversely as the square of the distance from it, we have for the illumination of the point D by light A $\frac{a}{x^2}$, and for the illumination of the same point by light B, $\frac{b}{(c-x)^2}$. But by the conditions of the problem these effects are equal; hence we have the equation to be discussed; viz.,

$$\frac{a}{x^2} = \frac{b}{(c-x)^2}.$$

This gives $$\frac{(c-x)^2}{x^2} = \frac{b}{a}; \text{ or } \frac{c-x}{x} = \pm\sqrt{\frac{b}{a}} = \frac{\pm\sqrt{b}}{\sqrt{a}};$$

or $$\frac{c}{x} - 1 = \frac{\pm\sqrt{b}}{\sqrt{a}}; \text{ or } \frac{c}{x} = \frac{\sqrt{a}\pm\sqrt{b}}{\sqrt{a}};$$

or, finally, $$x = c\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}, \text{ and } x = c\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}},$$

which are the values of x to be discussed.

Discussion.—I. *Let* c *be finite and* > 0.

1. When $a > b$, $x = c\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} > \frac{1}{2}c$, since $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} > \frac{1}{2}$ for $a > b$. This is as it should be, since for $a > b$ the point of equal illumination will evidently be nearer to B than to A. Again, the other value of x gives $x = c\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} > c$, since $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$ is + and > 1, when $a > b$. Hence we learn that there is a point beyond B, as at D′, where the illumination is the same from each light.

If we assume $\sqrt{a} = 2\sqrt{b}$, AD $= \frac{2}{3}c$, and AD′ $= 2c$.

2. It is evidently unnecessary to consider the case when $a < b$, since this would only situate the points of equal illumination with reference to A as the preceding discussion does with reference to B.

3. When $a=b$, $x=c\dfrac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}=\frac{1}{2}c$, since $\dfrac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}=\dfrac{\sqrt{a}}{2\sqrt{a}}=\frac{1}{2}$. This is as it should be, since it is evident that in this case the point of equal illumination is midway between the lights. Again, for the second value of x, we have $x=c\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}=\infty$. This is also evidently correct; for when the lights are of equal intensity there can be no point beyond B, for example, at which the illumination from A will be equal to that from B, except when $x=\infty$, for which the illumination is 0 for each light. [Let the student give the reason.]

II. *When* c = 0 In this case the original equation $\dfrac{a}{x^2}=\dfrac{b}{(c-x)^2}$ becomes $\dfrac{a}{x^2}=\dfrac{b}{x^2}$, whence $a=b$. We then have $x=c\dfrac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}=0$; and $x=c\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}=\dfrac{c\sqrt{a}}{\sqrt{a}-\sqrt{b}}=\dfrac{0}{0}$. The former shows that there is a point of equal illumination where the lights are (when $c=0$ they are together), and the latter shows that any point in the line is equally illuminated by each light. Both these conclusions are evidently correct.*

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* In discussing this problem, some have committed the error of considering that, since fo $c=0$ and a and b unequal, $x=c\dfrac{\sqrt{a}}{\sqrt{a}\pm\sqrt{b}}=0$, therefore there is a point of equal illumination at the point where the lights are situated! This is evidently absurd, since the hypothesis is that the lights are of *unequal* intensity. The error consists in not perceiving that the hypothesis, $c=0$, excludes the hypothesis, a and b unequal. That the hypotheses $a \gtrless b$ are excluded by the hypotheses $c=0$ and that there is a point of equal illumination, is self-evident. Perhaps the student may think that these conditions are no more inconsistent than those in I. 3, above, viz., c finite, $a=b$, and a point of equal illumination; and that, if in the former case we interpret $x=c\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}=\infty$ as indicating a point of equal illumination at $x=\infty$, we should in this interpret $x=\dfrac{c\sqrt{a}}{\sqrt{a}-\sqrt{b}}=0$ as indicating a point of equal illumination at the place where the lights are situated. But the closing remark in I. 3 will clear up this difficulty.

APPENDIX

SECTION I.

SERIES.

303. *A Series* is a succession of related quantities each of which, except the first or a certain number of the first, depends upon the next preceding, or a certain number of the next preceding, according to a common law. Each of the quantities is called a TERM OF THE SERIES.

ILL.—A Progression, as 1, 3, 5, 7, etc., or 3, 6, 12, 24, etc., is a series in which each term after the first depends upon the next preceding according to a common law. The numbers 1, 3, 7, 11, 21, 39, 71, 131, etc., constitute a series in which each term after the *third* is the *sum* of the *three* next preceding. The numbers 2, 3, 5, 17, 88, 1513, etc., constitute a series in which each term after the first three is the product of the *two* next preceding + the third preceding.

304. *A Recurring Series* is a series in which each term after the first n is equal to the sum of the products of each of the n preceding terms multiplied respectively by certain quantities which remain the same throughout the series. These multipliers with their respective signs constitute the ***Scale of Relation.***

ILL. $1, 4x, 9x^2, 16x^3$, etc., is a recurring series whose scale of relation is $x^3, -3x^2, 3x$, since $(1 \times x^3) + (4x \times [-3x^2]) + (9x^2 \times 3x) = 16x^3$. The next term after $16x^3$ would be $(4x \times x^3) + (9x^2 \times [-3x^2]) + (16x^3 \times 3x) = 25x^4$. The next would be $36x^5$.

305. *An Infinite Series* is one which has an infinite number of terms. Such a series is said to be *Convergent* when the successive terms decrease according to such a law as to make the sum finite; otherwise it is called *Divergent.*

ILL. $\frac{3}{10}, \frac{3}{100}, \frac{3}{1000}, \frac{3}{10000}$, etc., to infinity, is an infinite, converging series whose sum is $\frac{1}{3}$. That $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} +$ etc., to infinity * $= \frac{1}{3}$ is evident, since by division we have $\frac{1}{3} = .3333 + = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} +$ etc.

* The expression "to infinity" is usually omitted, as being sufficiently indicated by "etc.;" and, in fact, either the + sign at the end or the "etc." may be omitted.

306. To Revert a Series involving an unknown quantity is to express the value of that unknown quantity in terms of another quantity which is assumed as the sum of the first series, or as involved in that sum. Thus the general problem is, having given $f(y) = ax + bx^2 + cx^3 + \text{etc.}$, to express x in terms of y, *i. e.*, to find $x = f(y)$.

ILL.—Thus to revert the series $x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \text{etc.}$, is to express the value of x in another series involving y when $y = x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \text{etc.}$, or when $1 - 2y + 5y^3 = x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \text{etc.}$, etc.

307. The First Order of Differences of a series is the series of terms obtained by subtracting the 1st term of the given series from the 2d, the 2d from the 3d, the 3d from the 4th, etc. *The Second Order* is obtained from the first as the first is from the primitive series. *The Third Order* is obtained in like manner from the second; etc.

These several series are called the *Successive Orders of Differences.*

ILL.—Having the series

	1,		8,		27,		64,		125,	etc.,	we obtain
1st order of diff's,		7,		19,		37,		61,	etc.,		
2d " " "			12,		18,		24,	etc.,			
3d " " "				6,		6,	etc.,				
4th " " "					0,	etc.					

308. Interpolation is the process of finding intermediate functions between given non-consecutive functions of a series, without the labor of computing them from the fundamental formula of the series.

ILL.—The logarithms of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, etc., constitute a series of functions. Now knowing these, interpolation teaches how to find intermediate logarithms, as log 4.3, 4.5, 4.6, etc., or 2.7, 2.72, 3.102, 7.025, etc., without the labor of computing them from the fundamental formula of the series (***192***).

[NOTE.—The student must guard against the notion that every series is a recurring series. *Any succession of numbers related to each other by a common law*, as, for example, the logarithms of the natural numbers, is a series, as well as the more simple arithmetical, geometrical, and other recurring successions.]

309. Some of the more important problems concerning infinite series are: To find the scale of relation of a series; To find the nth (any) term of a series; To determine whether a series is convergent or divergent; To find the sum of a convergent series, or of n terms of any series; To revert a series; and, To interpolate terms between given terms. To these problems we shall give attention after having demonstrated the following lemma, which is of use in the solution of several of them.

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310. *Lemma.*—*The first term of the* n*th order of differences is* $a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)(n-2)}{\lfloor 3}d + \text{etc.}$, *when* n *is even, and* $-a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)(n-2)}{\lfloor 3}d - \text{etc.}$, *when* n *is odd;* a, b, c, d, etc., *being successive terms of the series.*

DEM.—Letting a, b, c, d, e, f, etc., be the series, we have

1st Order of diff's,	$b - a,\ c - b,\ d - c,\ e - d,\ f - e$, etc.,
2d " " "	$c - 2b + a,\ d - 2c + b,\ e - 2d + c,\ f - 2e + d$, etc.,
3d " " "	$d - 3c + 3b - a,\ e - 3d + 3c - b,\ f - 3e + 3d - c$, etc.,
4th " " "	$e - 4d + 6c - 4b + a,\ f - 4e + 6d - 4c + b$, etc.,
5th " " "	$f - 5e + 10d - 10c + 5b - a$, etc.

Now *by inspection* we observe that, numerically, the coefficients in these terms follow the law of the coefficients in the development of a binomial. Thus the coefficients in any term of the 2d order of differences, as in $c - 2b + a$, are the same as in the square of a binomial; those in any term of the 3d order, as in $d - 3c + 3b - a$, are the same as in the cube of a binomial, etc. Hence, reversing the order of the simple terms in the *first* terms of the successive orders, and representing the first term of the first order by D_1, the first term of the 2d order by D_2, the 1st term of the 3d order by D_3, etc., we have, for the *even* orders,

$$D_2 = a - 2b + c,$$
$$D_4 = a - 4b + 6c - 4d + e.$$

Hence, by induction, we have, for the 1st term of the nth order, when n is *even*,

$$D_n = a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)(n-2)}{\lfloor 3}d + \text{etc.}$$

Again, for the *odd* orders, we have

$$D_1 = -a + b,$$
$$D_3 = -a + 3b - 3c + d,$$
$$D_5 = -a + 5b - 10c + 10d - 5e + f.$$

Hence, by induction, when n is *odd*, the first term of the nth order is

$$D_n = -a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)(n-2)}{\lfloor 3}d - \text{etc.*}$$

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* The author does not deem it expedient to take the time and space to demonstrate more rigorously this law; nor does he fully sympathize with the idea that *induction* is in no case a satisfactory mathematical argument.

311. Cor.—*It will be observed that in order to find the* 1st *term of the first order of differences, we must have* 2 *terms of the series given; to find the* 1st *term of the* 2d *order,* 3 *terms; to find the* 1st *term of the* 3d *order,* 4 *terms; and, in general, to find the* 1st *term of the* n*th order we must know* n + 1 *terms of the series.*

EXAMPLES.

1. Find the 1st term of the 3d order of differences in the series 7, 12, 21, 36, 62, etc. Also the 1st term of the 4th order.

SUG'S.—For the 3d order we have

$$D_3 = -a + 3b - 3c + d = -7 + 3\cdot 12 - 3\cdot 21 + 36 = 2.$$

For the 4th order,

$$D_4 = a - 4b + 6c - 4d + e = 7 - 4\cdot 12 + 6\cdot 21 - 4\cdot 36 + 62 = 3.$$

2 to 6. Find the first terms of the orders of differences specified in the following:

(2.) 2d, 3d, and 4th, in 1, 8, 27, 64, 125, etc.
(3.) 3d, and 5th, in 1, 3, 3^2, 3^3, 3^4, 3^5, etc.
(4.) 5th, in 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc.
(5.) 5th, in 1, 6, 21, 56, 126, 252, etc.
(6.) 6th, in 3, 6, 11, 17, 24, 36, 50, etc.

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312. Prob. 1.—*To find the Scale of Relation in a recurring infinite series when a sufficient number of terms is given.*

SOLUTION.—1st. *When each term after the first depends on the next preceding.*—Let m represent the scale of relation. Then $b = ma$ (***304***). Whence $m = \frac{b}{a}$.

2d. *When each term after the first two depends on the two terms next preceding it.*—Letting m, n be the scale of relation, we have $c = ma + nb$, and $d = mb + nc$ (***304***). Whence $m = \frac{c^2 - bd}{ac - b^2}$, and $n = \frac{ad - bc}{ac - b^2}$.

3d. *When each term after the first three depends on the three terms next preceding it.*—Letting m, n, r represent the scale, we have $d = ma + nb + rc$, $e = mb + nc + rd$, and $f = mc + nd + re$. From these three equations the values of m, n, and r can be found.

4th. We can evidently proceed in a similar manner when the dependence is upon any number of preceding terms.

313. SCH.—In applying this method, if we assume that the dependence is upon more terms than it really is, one or more of the terms of the scale will reduce to 0. If we assume the dependence to be on too few terms, the error will appear in attempting to apply the scale when found. If we attempt to apply the method to a series which is not recurring, the error will appear in the form which the scale assumes, or when we attempt to apply it.

When the dependence is upon two terms, any two equations of the series $c = ma + nb$, $d = mb + nc$, $e = mc + nd$, $f = md + ne$, etc., will give the *same values* for m and n. So also if the dependence is upon three terms, any three equations of the series $d = ma + nb + rc$, $e = mb + nc + rd$, $f = mc + nd + re$, $g = md + ne + rf$, etc., will give the *same values* to m, n, and r; etc., etc.

There is no general method of determining that a series is absolutely *not* recurring. The best practical method of procedure is to assume *first* that the dependence is upon *two* terms: if this does not give a scale which will extend the series, try whether the dependence is not upon three terms, then upon four, etc. Of course, applying this process to an infinite series would not determine that the series was absolutely *not* recurring.

EXAMPLES.

1. Find the scale of relation in the series 1, 12, 48, 384, 1920, etc.

SUG'S.—Assuming that the dependence is upon two terms, we have $48 = m + 12n$, and $384 = 12m + 48n$; whence $m = 24$, and $n = 2$. Now since $1920 = 24 \cdot 48 + 2 \cdot 384$, we conclude that $+24$, $+2$, is the scale.

2. Find the scale of relation in the series $1, 6x, 12x^2, 48x^3, 120x^4$, etc.

SUG'S.—We have $12x^2 = m + 6xn$, and $48x^3 = 6xm + 12x^2n$; whence $m = 6x^2$, and $n = x$. Now, as $120x^4 = 6x^2 \cdot 12x^2 + x \cdot 48x^3$, we conclude that the scale of relation is $+6x^2$, $+x$.

3. Find the scale of relation in the series $1, 4x, 6x^2, 11x^3, 28x^4, 63x^5$, and extend the series two terms.

Scale of relation, $+3x^3, -x^2, +2x$.
Next two terms, $131x^6, 283x^7$.

4 to 11. Find the scale of relation in the following, and extend each series 2 or 3 terms:

(4.) $1, x, 2x^2, 2x^3, 3x^4, 3x^5, 4x^6, 4x^7$, etc.
(5.) 1, 3, 18, 54, 243, 729, 2916, 8748, etc.
(6.) $1, x, 5x^2, 13x^3, 41x^4, 121x^5, 365x^6$, etc.
(7.) 1, 4, 12, 32, 80, etc.
(8.) $3, 5x, 7x^2, 13x^3, 23x^4, 45x^5$, etc.
(9.) $\frac{a}{b}, -\frac{ac}{b^2}x, \frac{ac^2}{b^3}x^2, -\frac{ac^3}{b^4}x^3$, etc.

(10.) 1, 4, 10, 20, 35, 56 84, 120, etc.
(11.) 1, 4, 8, 13, 19, 26, 34, etc.

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314. Prob. 2.—*To find the* n*th term of a series when a sufficient number of terms is given.*

Solution.—The best method of doing this depends upon the character of the series. We give the following:

1st. The formula $l = a + (n-1)d$, and $l = ar^{n-1}$, resolve the problem for arithmetical and geometrical series, l being any term.

2d. The scale of relation may be determined by Prob. 1, and the series extended to the nth term by means of it.

3d. But the first terms of the successive orders of differences afford one of the most elegant and general methods. Thus from (*310*) we have

$D_1 = -a+b$; $\quad\therefore b = a + D_1$;
$D_2 = a - 2b + c$; $\quad\therefore c = a + 2D_1 + D_2$; *
$D_3 = -a + 3b - 3c + d$; $\quad\therefore d = a + 3D_1 + 3D_2 + D_3$; †
$D_4 = a + 4b - 6c + 4d - e$; $\quad\therefore e = a + 4D_1 + 6D_2 + 4D_3 + D_4$;
$D_5 = -a + 5b - 10c + 10d - 5e + f$; $\quad\therefore f = a + 5D_1 + 10D_2 + 10D_3 + 5D_4 + D_5$.
etc., etc., etc.

Whence, by induction, we have, in general, the nth term $= a + (n-1)D_1 + \frac{(n-1)(n-2)}{2}D_2 + \frac{(n-1)(n-2)(n-3)}{\underline{|3}}D_3 +$ etc., till the term containing D_{n-1} is reached, or till an order of differences is reached of which each term is 0. It is only in the latter case that the method is practically useful, since to determine the first terms of the $n-1$ successive orders of differences, requires that n terms of the series be known.

Examples.

1 to 5. Solve the following by means of the scale of relation:

(1.) Find the 8th term of 1, $2x$, $8x^2$, $28x^3$, $100x^4$, etc.
(2.) Find the 9th term of 1, $3x$, $5x^2$, $7x^3$, $9x^4$, $11x^5$, etc.
(3.) Find the 10th term of 1, $3x$, $2x^2$, $-x^3$, $-3x^4$, $-2x^5$, etc.
(4.) Find the 12th term of 3, 5, 7, 13, 23, 45, etc.
(5.) Find the 11th term of 1, 1, 5, 13, 41, 121, etc.

6 to 12. Solve the following by means of the successive orders of differences:

(6.) The 12th term of 1, 5, 15, 35, 70, 126, etc.

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* $c = -a + 2b + D_2 = -a + 2(a + D_1) + D_2 = a + 2D_1 + D_2$.

† $d = a - 3b + 3c + D_3 = a - 3(a + D_1) + 3(a + 2D_1 + D_2) + D_3 = a + 3D_1 + 3D_2 + D_3$.

(7.) The 15th term of 1, 3, 6, 10, 15, 21, etc. Also the nth.

(8.) The nth term of $1\cdot 2$, $2\cdot 3$, $3\cdot 4$, $4\cdot 5$, etc.

(9.) The 12th term of 1, $4x$, $6x^2$, $11x^3$, $28x^4$, $63x^5$, etc.*

(10.) Solve the first five given above by this method, when it will apply. Also determine the scale of relation in (6) to (9) in cases in which the series is recurring.

(11.) Find the nth term of 1, 2^2, 3^2, 4^2, etc.

(12.) Find the 9th term of 70, 252, 594, 1144, 1950, etc.

13. Extend the following to 10 terms by the method of differences: 1, 4, 8, 13, 19, etc. Also x^2, $4x^4$, $8x^6$, $13x^8$, $19x^{10}$, etc. Also 1, 6, 20, 50, 105, 196, etc,

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315. Prob. 3.—*To determine whether a series is convergent or divergent.*

SOLUTION.—1st. *When the terms are all* +. If the series is not decreasing, of course it cannot be convergent. Thus $a+b+c+d+e+$ etc., if $a<b<c<d<e$, etc., is $> a\infty$. Let us then consider the case when the terms are all +, and $a>b>c>d>e$, etc. We have

$$S=a+b+c+d+e+\text{etc.}=a\left(1+\frac{b}{a}+\frac{c}{a}+\frac{d}{a}+\frac{e}{a}+\text{etc.}\right)$$

$$=a\left(1+\frac{b}{a}+\frac{cb}{ba}+\frac{dcb}{cba}+\frac{edcb}{dcba}+\text{etc.}\right).$$

Now if $\frac{b}{a}$, $\frac{c}{b}$, $\frac{d}{c}$, $\frac{e}{d}$, etc. $<p$, $S<a(1+p+p^2+p^3+p^4+\text{etc.})$, which, if $p<1$, $=\frac{a}{1-p}$. Therefore, *An infinite series of positive terms is always convergent, if the ratio of each term to the preceding term is less than some assignable quantity which is itself less than* 1.

2d. *When the terms are alternately + and −, and decreasing.* Let the series be a, $-b$, $+c$, $-d$, $+e$, $-$ etc. Now we may write

$$S=(a-b)+(c-d)+(e-f)+\text{etc.};$$

and also

$$S=a-(b-c)-(d-e)-\text{etc.}$$

Since the terms are decreasing $(c-d)$, $(e-f)$, etc., are +, and $S>a-b$. Again, $(b-c)$, $(d-e)$, etc., are +, and $S<a$. Therefore, *Any series of decreasing terms, which terms are alternately + and −, is convergent.*

3d. *When the terms are alternately + and −, and increasing,* we have

$$S=a-b+c-d+e-f+g-\text{etc.}=a-(b-c)-(d-e)-(f-g)-\text{etc.}$$

Now, since the terms are increasing, $b-c$, $d-e$, $f-g$, etc., are essentially negative. Representing these differences by $-d$, $-d_1$, $-d_2$, etc., we have

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* It is evident that the 12th term involves x to the 11th power, or contains x^{11}. Hence we have only to find the coefficient, or the 12th term of the series 1, 4, 6, 11, 28, 63, etc.

$S = a + d + d_1 + d_2 +$ etc., a series which can be examined by the first process given above.

4th. The process of grouping the terms and thus forming a new series, as in the last case, is frequently serviceable in other cases than that there specified.*

EXAMPLES.

1. Determine whether $1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} +$ etc., is a convergent series.

SUG'S.—Here $\frac{b}{a} = 1$, $\frac{c}{b} = \frac{1}{2}$, $\frac{d}{c} = \frac{1}{3}$, $\frac{e}{d} = \frac{1}{4}$, etc.; whence we see that each of the ratios after the second is less than $\frac{1}{2}$, which is itself less than 1. Hence the series is converging.

2. Determine whether $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} +$ etc., is a converging series.

3 to 6. Determine which of the following are converging:

(3.) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$ etc.

(4.) $1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} +$ etc., r being > 1, *i. e.*, any decreasing geometrical progression.

(5.) $\frac{1}{8 \cdot 18} + \frac{1}{10 \cdot 21} + \frac{1}{12 \cdot 24} + \frac{1}{14 \cdot 27} +$ etc.

(6.) $\frac{3}{1 \cdot 2 \cdot 2} + \frac{4}{2 \cdot 3 \cdot 2^2} + \frac{5}{3 \cdot 4 \cdot 2^3} +$ etc.

7. For what values of x is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} +$ etc., convergent, and for what divergent?

SUG'S.—For $x \overline{\lessdot} 1$ we have a series with the terms alternately + and −, and decreasing. Hence, by (*315*, 2d), the series is convergent. Again, to examine the series for $x > 1$, it may be written $x - \frac{x^2}{2} + x^3\left(\frac{1}{3} - \frac{x}{4}\right) + x^5\left(\frac{1}{5} - \frac{x}{6}\right) + x^7\left(\frac{1}{7} - \frac{x}{8}\right) +$ etc. Now, for $x > 1$, some one of the factors $\left(\frac{1}{3} - \frac{x}{4}\right)$, $\left(\frac{1}{5} - \frac{x}{6}\right)$, $\left(\frac{1}{7} - \frac{x}{8}\right)$, etc., and all following it will become negative. Thus, if $x = \frac{8}{7}$, all following $\frac{1}{7} - \frac{x}{8}$ will be negative.

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* This is confessedly quite an imperfect presentation of this problem; but it is sufficient for most purposes, and is as full as our limits will allow.

The sum of that portion of the series preceding this first negative factor will be finite, since it will be composed of a finite number of finite terms. Let us now examine the infinite series which is composed of negative terms. Let a be the value of x for which we are examining the series, and y the exponent of x in the first negative term. This term is therefore $a^y\left(\frac{1}{y} - \frac{a}{y+1}\right)$. Now this may be taken as the general term of this portion of the series if we understand that a is constant and y variable. As y increases by 2 in each successive term, the first two terms of this series are $a^y\left(\frac{1}{y} - \frac{a}{y+1}\right)$, $a^{y+2}\left(\frac{1}{y+2} - \frac{a}{y+3}\right)$; and the ratio of the second to the first is $a^2\left\{\frac{y+3-ay-2a}{(y+2)(y+3)} \times \frac{y(y+1)}{y+1-ay}\right\}$ $= a^2\left\{\frac{(1-a)y^3-(3a-4)y^2-(2a-3)y}{(1-a)y^3+(6-5a)y^2+(11-6a)y+6}\right\}$, the limit of which, as y increases to infinity, is a^2. But as $a > 1$, $a^2 > 1$, and this negative series is divergent and its sum is infinite. Hence the given series is convergent for $x \lesseqgtr 1$, and for all values of $x > 1$ it is divergent.

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316. Prob. 4.—*To find the sum of* n *terms of a series.*

This problem, like many others concerning series, does not admit of a general solution. We specify the following cases:

CASE 1.—*When the series is* ARITHMETICAL *or* GEOMETRICAL, *either divergent or convergent,* for n finite, $S = \frac{1}{2}n[2a + (n-1)d]$, or $S = \frac{a(r^n - 1)}{r-1}$. For an infinite geometrical convergent series we have $S = \frac{a}{1-r}$.

CASE 2.—*When the series is an infinite, decreasing,* RECURRING *series,* to find the sum of the series (*i. e.*, n being ∞). Let the series be $a+b+c+d+e+$ etc., and m, n the scale of relation, the dependence being upon two terms. Whence we have

$$\begin{aligned} a &= a, \\ b &= b, \\ c &= am + bn, \\ d &= bm + cn, \\ e &= cm + dn, \\ f &= dm + en, \\ &\cdots\cdots \end{aligned}$$

Putting $S = a+b+c+d+$ etc., $\cdots\cdots$

and adding, this gives $S = a + b + Sm + (S - a)n.$

Solving for S, we have $$S = \frac{a + b - an}{1 - m - n}. \quad (1)$$

When the scale of relation consists of three terms, as m, n, r, we have

$$\begin{aligned} a &= a, \\ b &= b, \\ c &= c, \\ d &= am + bn + cr, \\ e &= bm + cn + dr, \\ f &= cm + dn + er, \\ g &= dm + en + fr, \\ &\cdots \\ &\cdots \end{aligned}$$

Whence $\quad S = a + b + c + Sm + (S - a)n + (S - a - b)r.$

And solving for S, $\quad S = \dfrac{a + b + c - an - (a + b)r}{1 - m - n - r}$. (2)

When the scale of relation consists of four terms, as m, n, r, s, we can write from analogy,

$$S = \frac{a + b + c + d - an - (a + b)r - (a + b + c)s}{1 - m - n - r - s}. \qquad (3)$$

CASE 3.—*To find the sum of* n *terms of a series by the method of differences.*—Let the series be a, b, c, d, e, f, etc., which we will call (A).

Now if we write the series

$(B) \quad 0, a, a + b, \ a + b + c, \ a + b + c + d, \ a + b + c + d + e$, etc.,

of which the series (A) is the first order of differences, it is evident that the $(n + 1)$th term of (B) is the sum of n terms of the given series (A). By the formula for the nth term (**314**, 3d), which is

The nth term $= a + (n-1)D_1 + \dfrac{(n-1)(n-2)}{2} D_2 + \dfrac{(n-1)(n-2)(n-3)}{\underline{|3}} D_3 +$ etc.,

noticing that a, the first term, in series (B) is 0, that D_1 of series (B) is a of series (A), D_2 of series (B) is D_1 of series (A), etc., we have, for the sum of n terms of (A)

$$S = na + \frac{n(n-1)}{2} D_1 + \frac{n(n-1)(n-2)}{\underline{|3}} D_2 + \text{etc.}$$

On this formula we observe that when the orders of differences do not vanish, if the series is extended to the $(n+1)$th term the coefficient of that term will become 0, and the series will terminate.

Moreover, in cases in which the nth order of differences vanishes, the same number of terms of this formula will give the sum of any number of terms of the series above the nth.

CASE 4.—Upon the principle that any fraction of the form $\dfrac{q}{n(n + p)}$ $= \dfrac{1}{p}\left(\dfrac{q}{n} - \dfrac{q}{n + p}\right)$,* many series of fractional terms of the form $\dfrac{q}{n(n + p)}$ may be summed.

* This is evident since $\dfrac{q}{n} - \dfrac{q}{n+p} = \dfrac{nq + pq - nq}{n(n+p)} = \dfrac{pq}{n(n + p)}$.

Also many series of fractional terms of the form $\dfrac{q}{n(n+p)(n+2p)}$ may be summed from the fact that

$$\frac{q}{n(n+p)(n+2p)} = \frac{1}{2p}\left\{\frac{q}{n(n+p)} - \frac{q}{(n+p)(n+2p)}\right\}.$$

When the fractional terms are of the form $\dfrac{q}{n(n+p)(n+2p)(n+3p)}$, the summation may often be effected upon the principle that

$$\frac{q}{n(n+p)(n+2p)(n+3p)} = \frac{1}{3p}\left\{\frac{q}{n(n+p)(n+2p)} - \frac{q}{(n+p)(n+2p)(n+3p)}\right\}.$$

The practicability of this method depends upon our ability to find the difference between two series. Thus, when the terms of the given series are of the form $\dfrac{q}{n(n+p)}$, if we can find the difference between two series whose terms are of the form $\dfrac{q}{n}$, and $\dfrac{q}{n+p}$ respectively, we can find the sum of the given series. But the method will be more readily comprehended in connection with its application. (See Ex's 15–30.)

Examples.

1 to 7. Find the sum of the following recurring series:

(1.) $1 + 2x + 8x^2 + 28x^3 + 100x^4 + \text{etc.}$

(2.) $1 + 2x + 3x^2 + 5x^3 + 8x^4 + \text{etc.}$

(3.) $1 + 3x + 5x^2 + 7x^3 + \text{etc.}$

(4.) $3 + 5x + 7x^2 + 13x^3 + 23x^4 + 45x^5 + \text{etc.}$

(5.) $1 + 1 + 5 + 13 + 41 + 121 + \text{etc.}$

(6.) $1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \text{etc.}$

(7.) $\dfrac{a}{b} - \dfrac{ac}{b^2}x + \dfrac{ac^2}{b^3}x^2 - \dfrac{ac^3}{b^4}x^3 + \text{etc.}$

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8 to 14. Find the sum of the following by the method of differences:

(8.) $1+3+5+7+\text{etc.}$, to 20 terms; to n terms.

(9.) $1+2+3+4+5+\text{etc.}$, to 50 terms; to n terms.

(10.) $1+5+15+35+70+126+\text{etc.}$, to 30 terms; to n terms.

(11.) $70+252+594+1144+1950$ etc., to 25 terms; to n terms.

(12.) $1+2^4+3^4+4^4+\text{etc.}$, to 12 terms; to n terms.

(13.) $1+2^2+3^2+4^2+\text{etc.}$, to n terms.

(14.) $1+2^3+3^3+4^3+\text{etc.}$, to n terms.

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15. Find the sum of $\dfrac{1}{1\cdot2} + \dfrac{1}{2\cdot3} + \dfrac{1}{3\cdot4} + \dfrac{1}{4\cdot5} + \text{etc.}$, by the method given in Case 4.

SUG'S.—If we put $p=1$, $q=1$, and $n=1, 2, 3, 4$, etc., successively, the general form of the term in this series is $\frac{q}{n(n+p)}$. Thus we have

For the 1st term, $\frac{q}{n(n+p)}=\frac{1}{1(1+1)}=\frac{1}{1}\left(\frac{1}{1}-\frac{1}{1+1}\right)^*=1\left(1-\frac{1}{2}\right)$;

For the 2d term, $\frac{q}{n(n+p)}=\frac{1}{2(2+1)}=\frac{1}{1}\left(\frac{1}{2}-\frac{1}{2+1}\right)^*=1\left(\frac{1}{2}-\frac{1}{3}\right)$;

For the 3d term, $\frac{q}{n(n+p)}=\frac{1}{3(3+1)}=\frac{1}{1}\left(\frac{1}{3}-\frac{1}{3+1}\right)^*=1\left(\frac{1}{3}-\frac{1}{4}\right)$;

For the 4th term, $\frac{q}{n(n+p)}=\frac{1}{4(4+1)}=\frac{1}{1}\left(\frac{1}{4}-\frac{1}{4+1}\right)^*=1\left(\frac{1}{4}-\frac{1}{5}\right)$;

etc., etc., etc.

Putting S for the sum of the series and adding, we have

$$S=(1-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+(\tfrac{1}{4}-\tfrac{1}{5})+\text{etc.}$$

$$=\left\{\begin{array}{l}1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{etc.}\\ \quad-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\text{etc.}\end{array}\right\}=1.$$

NOTE.—It will be seen that this method is only an ingenious device for decomposing the given infinite series into two infinite series, one of which destroys all but a finite portion of the other.

16. Find the sum of $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+$ etc.

17. Find the sum of n terms of each of the two preceding series.

SUG.—We have for the nth term of the last series $q=1$, $p=2$, $n=2n-1$, since $2n-1$ is the nth odd number. Hence for the nth term $\frac{q}{n(n+p)}$ $=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$. We therefore have

$$S=\frac{1}{2}\left\{\begin{array}{l}1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\ \cdots\cdots\ \frac{1}{2n-1}\\ \quad-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}\ \cdots\cdots\ \frac{1}{2n-1}-\frac{1}{2n+1}\end{array}\right\}=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$$

$$=\frac{n}{2n+1}.$$

18. Find the sum of $\frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\frac{1}{4\cdot 7}+$ etc. Also of n terms of the same.

19. Find the sum of $\frac{2}{15}-\frac{3}{35}+\frac{4}{63}-\frac{5}{99}+$ etc., to n terms.

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* Since by Case 4, $\frac{q}{n(n+p)}=\frac{1}{p}\left(\frac{q}{n}-\frac{q}{n+p}\right)$.

SUG'S.—It will be seen that this series is the same as $\frac{2}{3\cdot 5}-\frac{3}{5\cdot 7}+\frac{4}{7\cdot 9}-\frac{5}{9\cdot 11}$ + etc. Hence by making $q=2, 3, 4, 5$, etc., successively, $n=3, 5, 7, 9$, etc., successively, and $p=2$, we have

$$\tfrac{1}{2}\{(\tfrac{2}{3}-\tfrac{2}{5})-(\tfrac{3}{5}-\tfrac{3}{7})+(\tfrac{4}{7}-\tfrac{4}{9})-(\tfrac{5}{9}-\tfrac{5}{11})+\text{etc.}\},\quad\text{or}$$
$$\tfrac{1}{2}\{\tfrac{2}{3}-(\tfrac{2}{5}+\tfrac{3}{5})+(\tfrac{3}{7}+\tfrac{4}{7})-(\tfrac{4}{9}+\tfrac{5}{9})+\tfrac{5}{11}+\text{etc.}\}$$
$$=\tfrac{1}{2}\{\tfrac{2}{3}-1+1-1+\tfrac{5}{11}+\text{etc.}\}.$$

Now the form of this last term is $\frac{n+1}{2n+3}$; and if an even number of terms of the given series is taken, we have $\frac{1}{2}\left\{\frac{2}{3}-1+\frac{n+1}{2n+3}\right\}$, all the intermediate terms destroying each other. But if an odd number is taken, we have $\frac{1}{2}\left(\frac{2}{3}-\frac{n+1}{2n+3}\right)$. Finally, as $\frac{n+1}{2n+3}=\frac{1}{2}-\frac{1}{2(2n+3)}$, we have for an even number of terms $\frac{1}{2}\left\{\frac{2}{3}-\frac{1}{2}-\frac{1}{2(2n+3)}\right\}$, or $\frac{1}{12}-\frac{1}{4(2n+3)}$; and for an odd number, $\frac{1}{2}\left\{\frac{2}{3}-\frac{1}{2}+\frac{1}{2(2n+3)}\right\}$, or $\frac{1}{12}+\frac{1}{4(2n+3)}$. When $n=\infty$, we have $\frac{1}{4(2n+3)}=0$; whence the sum is $\frac{1}{12}$.

20. Find the sum of $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+$ etc.

21. Find the sum of $\frac{1}{1\cdot 3}-\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}-$ etc.

22. Find the sum of $\frac{1}{3\cdot 8}+\frac{1}{6\cdot 12}+\frac{1}{9\cdot 16}+$ etc.

SUG.—This equals $\frac{1}{12}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\text{etc.}\right)$.

23. Find the sum of $\frac{4}{1\cdot 5}+\frac{4}{5\cdot 9}+\frac{4}{9\cdot 13}+\frac{4}{13\cdot 17}+$ etc.

24. Find the sum of $\frac{4}{1\cdot 2\cdot 3}+\frac{5}{2\cdot 3\cdot 4}+\frac{6}{3\cdot 4\cdot 5}+$ etc.

SUG'S.—By putting $p=1$, $q=4, 5, 6$, etc., successively, and $n=1, 2, 3$, etc., successively, these terms take the form $\frac{q}{n(n+p)(n+2p)}$, and since $\frac{q}{n(n+p)(n+2p)}=\frac{1}{2p}\left\{\frac{q}{n(n+p)}-\frac{q}{(n+p)(n+2p)}\right\}$, we may write the given series thus:

$$\frac{1}{2}\left\{\left(\frac{4}{1\cdot 2}-\frac{4}{2\cdot 3}\right)+\left(\frac{5}{2\cdot 3}-\frac{5}{3\cdot 4}\right)+\left(\frac{6}{3\cdot 4}-\frac{6}{4\cdot 5}\right)+\text{etc.}\right\}$$

$$= \frac{1}{2}\left\{\begin{array}{l} \dfrac{4}{1\cdot 2} + \dfrac{5}{2\cdot 3} + \dfrac{6}{3\cdot 4} + \text{etc.} \\ \qquad - \dfrac{4}{2\cdot 3} - \dfrac{5}{3\cdot 4} - \text{etc.} \end{array}\right\}$$

$$= \frac{1}{2}\left(2 + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \text{etc.}\right) = \frac{1}{2}\left(2 + \frac{1}{2}\right) \text{ (see Ex. 15)} = 1\tfrac{1}{4}.$$

25. Find the sum of $\frac{3}{5\cdot 8\cdot 11} + \frac{9}{8\cdot 11\cdot 14} + \frac{15}{11\cdot 14\cdot 17} + \text{etc.}$

26. Find the sum of $\frac{1}{1\cdot 3\cdot 5} + \frac{2}{3\cdot 5\cdot 7} + \frac{3}{5\cdot 7\cdot 9} + \text{etc.}$

27. Find the sum of $\frac{1}{1\cdot 3\cdot 5} + \frac{4}{3\cdot 5\cdot 7} + \frac{7}{5\cdot 7\cdot 9} + \frac{10}{7\cdot 9\cdot 11} + \text{etc.}$

28. Find the sum of $\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \frac{1}{3\cdot 4\cdot 5\cdot 6} + \text{etc.}$

SUG.—Consider that $\frac{q}{n(n+p)(n+2p)(n+3p)} = \frac{1}{3p}\left\{\frac{q}{n(n+p)(n+2p)} - \frac{q}{(n+p)(n+2p)(n+3p)}\right\}.$

29. Find the sum of $\frac{1}{1\cdot 3\cdot 5\cdot 7} + \frac{2}{3\cdot 5\cdot 7\cdot 9} + \frac{3}{5\cdot 7\cdot 9\cdot 11} + \text{etc.}$

30. Find the sum of $\frac{2}{3\cdot 6\cdot 9\cdot 12} + \frac{5}{6\cdot 9\cdot 12\cdot 15} + \frac{8}{9\cdot 12\cdot 15\cdot 18} + \text{etc.}$

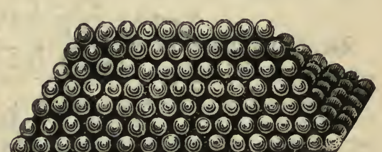
NOTE.—The above examples are taken from YOUNG'S ALGEBRA, an excellent old English work to which American editors are much indebted.

PILING BALLS AND SHELLS.

317. In arsenals and navy-yards, cannon-balls and shells are piled on a level surface in neat and orderly piles of three different forms, viz., *triangular, square,* and *oblong.* The figures below will sufficiently illustrate these forms:

TRIANGULAR PILE.

SQUARE PILE.



OBLONG PILE.

318. Prop.—*The formula for the number of balls or shells in a triangular pile having* n *balls or shells on a side of its lowest course is*

$$\tfrac{1}{6}n(n+1)(n+2).$$

DEM.—The student will be able to discover that, beginning at the top, the number of balls or shells in each course is as follows :

1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, - - - - etc.,

or 1, 3, 6, 10, 15, 21, - - - - etc.

Summing this series to n terms by the method of differences he will obtain the formula.

319. COR.—*The number of courses in a triangular pile is equal to the number of balls or shells in one side of the lowest course; and the number of balls or shells in the lowest course is* $1 + 2 + 3 + 4$ - - - - n, *or* $\frac{1}{2}(n^2 + n)$.

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320. Prop.—*The formula for the number of balls or shells in a square pile having* n *balls or shells on a side of its lowest course is*

$$\tfrac{1}{6}n\,(n + 1)\,(2n + 1).$$

The student should be able to demonstrate this as above.

321. COR.—*The number of courses in a square pile is equal to the number of balls or shells in one side of the lowest course; and the number of balls or shells in the lowest course is* $1 + 3 + 5 + 7 + 9$ - - - - $2n - 1$, *or* n^2.

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322. Prop.—*The formula for the number of balls or shells in an oblong pile having* m *balls or shells in the length of the base and* n *in the width is*

$$\tfrac{1}{6}n\,(n + 1)\,(3m - n + 1).$$

DEM.—Observe that there are as many courses as there are balls in the width of the base. Let m' be the number in the top row, whence we have for the number in the successive rows from the top downward,

$$m',\ 2(m' + 1),\ 3(m' + 2),\ 4(m' + 3),\ 5(m' + 4),\ \text{etc.}$$

Taking the successive differences, we find $D_1 = m' + 2$, $D_2 = 2$, and $D_3 = 0$. Substituting in

$$S = na + \frac{n(n-1)}{2}D_1 + \frac{n(n-1)(n-2)}{2\cdot 3}D_2,$$

we have $$S = m'n + \frac{n(n-1)}{2}(m' + 2) + \frac{n(n-1)(n-2)}{3},$$ which readily reduces to $$S = \tfrac{1}{6}n\{(n+1)(3m' + 2n - 2)\}.$$

Now m being the number of balls or shells in the length of the base, we observe that $m' = m - n + 1$, which substituted in the previous equation gives

$$S = \tfrac{1}{6}n(n + 1)(3m - n + 1).$$

SCH.—If we make $m = n$, this gives the formula for the square pile, as it should.

Examples.

1. Find the number of balls in a triangular pile of 20 courses. In a triangular pile with 42 balls on one side of the lowest course. How many balls in the bottom course? How many in one of the faces?

2. Find the number of shells in a square pile with 30 courses. With 23 balls in one side of the lowest course. With 2209 in the bottom course. How many balls in one face of each pile?

3. Find the number of balls in an oblong pile whose bottom course is 42 balls by 20. Whose top course contains 23 balls, and which has fifteen courses.

4. How many shells remain in an incomplete triangular pile whose top course contains 28 shells, and whose bottom course has 15 shells on a side?

5. How many balls in an incomplete square pile whose top course is 8 balls on a side, and whose bottom course is 20 balls on a side?

6. How many shells in an incomplete oblong pile whose top course is 12 by 20, and whose bottom course is 52 shells in length?

Reversion of Series.

323. Prob.*—To revert a Series.*

Solution.—The problem is, having given

$$f(y) = ax + bx^2 + cx^3 + dx^4 + \text{etc.,} \qquad (A)$$

to express x as a function of y, *i. e.*, to obtain

$$x = Ay + By^2 + Cy^3 + Dy^4 + \text{etc.,} \qquad (B)$$

the essential thing in the solution being to find the values of the indeterminate coefficients A, B, C, D, etc. To do this, we form x^2, x^3, x^4, etc., from (B) in terms of y, and substitute in the second member of (A). Whence we have $f(y) = f'(y)$.* From this relation we can obtain the values of the indeterminates A, B, C, D, etc., in the ordinary way.

Examples.

1. Given $y = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \text{etc.}$, to revert the series, *i. e.*, to express the value of x in a series involving y.

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* This notation means that both members are functions of y, but that they are not the same function: one is the f function, and the other the f' function.

SUG'S.—Assume $x = Ay + By^2 + Cy^3 + Dy^4 + \text{etc.}$

Whence $x^2 = A^2y^2 + 2ABy^3 + \left.\begin{matrix}2AC \\ + B^2\end{matrix}\right| y^4 + \text{etc.},$

$x^3 = A^3y^3 + 3A^2By^4 + \text{etc.},$

and $x^4 = A^4y^4 + \text{etc.},$ these developments being extended as far as is necessary in order to determine four terms of the reverted series.

Substituting these values in the given series we have

$$y = Ay + \left.\begin{matrix}B \\ +\frac{1}{2}A^2\end{matrix}\right| y^2 + \left.\begin{matrix}C \\ +AB \\ +\frac{1}{3}A^3\end{matrix}\right| y^3 + \left.\begin{matrix}D \\ +AC \\ +\frac{1}{2}B^2 \\ +A^2B \\ +\frac{1}{4}A^4\end{matrix}\right| y^4 + \text{etc.}$$

Whence $A = 1$, $B + \frac{1}{2}A^2 = 0$, $C + AB + \frac{1}{3}A^3 = 0$, and $D + AC + \frac{1}{2}B^2 + A^2B + \frac{1}{4}A^4 = 0$. These give $A = 1$, $B = -\frac{1}{2}$, $C = \frac{1}{6}$, and $D = -\frac{1}{24}$. Therefore the reverted series is $x = y - \frac{1}{2}y^2 + \frac{1}{\underline{|3}}y^3 - \frac{1}{\underline{|4}}y^4 + \text{etc.}$

2 to 6. Revert the following:

(2.) $y = x + x^2 + x^3 + x^4 + \text{etc.}$

(3.) $y = x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \text{etc.}$

(4.) $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \text{etc.}$*

(5.) $y = 2x + 3x^3 + 4x^5 + 5x^7 + \text{etc.}$

(6.) $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{\underline{|3}}x^3 + \frac{1}{\underline{|4}}x^4 + \text{etc.}$†

7. Required to express the value of y in terms of x from the relation

$$y + ay^2 + by^3 + cy^4 + \text{etc.} = mx + nx^2 + px^3 + qx^4 + \text{etc.}$$

INTERPOLATION.

324. Prob.—*Having given a series of functions* a, b, c, d, e, *etc., to find a function intermediate between any two of this series, which function shall conform to the law of the series.*

ILL.—Let the series of functions be the logarithms of 232, 233, 234, 235, etc., viz., 2.365488, 2.367356, 2.339216, and 2.371068; let it be required to find the logarithm of 233.4, *i. e.*, the function $\frac{2}{5}$ of the way from 2.367356 to 2.369216.

SOLUTION.—The solution of this problem is simply an application of the

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* In this example it will be more expeditious to assume $x = Ay + By^3 + Cy^5 + \text{etc.}$, though it is not essential.

† Transpose the 1, put $z = y - 1$, and then revert the series $z = x + \frac{1}{2}x^2 + \frac{1}{\underline{|3}}x^3 + \frac{1}{\underline{|4}}x^4 + \text{etc.}$ This is necessary, since the theory of Indeterminate Coefficients assumes that both variables become 0 at the same time; *i. e.*, that $x=0$, makes $z=0$.

formula for finding the nth term of a series by the Method of Differences (*314*); viz.,

$$\text{The } n\text{th term} = a + (n-1)D_1 + \frac{(n-1)(n-2)}{2}D_2 + \frac{(n-1)(n-2)(n-3)}{\lfloor 3}D_3 + \text{etc.}$$

But for our present purpose it is more convenient to replace the $(n-1)$ of the formula, where n represents the number of the term sought, by $\frac{p}{q}$, a fraction which indicates *the distance* of the term sought, from the first term used, this distance being measured by calling the distance between any two given terms 1. Thus in the series a, b, c, d, e, etc., a term $\frac{2}{3}$ of the way from b to c, would be reckoned at a distance $1\frac{2}{3}$, or $\frac{5}{3}$ from a, *i. e.*, $\frac{p}{q}$ would be $\frac{5}{3}$ in this case. Now by this method of reckoning it is evident that the $(n-1)$ of the formula must be replaced by $\frac{p}{q}$, for n stands for the number of the term, which is one more than the number of intervals between it and the first term. Thus the 4th term is 3 intervals from the first term. Making this substitution of $\frac{p}{q}$ for $n-1$ the formula becomes

$$\text{Term to be interpolated} = a + \frac{p}{q}D_1 + \frac{1}{2}\frac{p}{q}\left(\frac{p}{q}-1\right)D_2 + \frac{1}{6}\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)D_3 + \text{etc.}$$

325. SCH. 1.—On this formula we observe that when the series of functions is such that the differences vanish, *i. e.*, D_2, D_3, D_4, or some order becomes 0, the formula gives an absolutely correct result. But when the differences do not vanish, the result is only an approximation. However, such is the closeness of approximation, that for practical purposes only second differences are usually needed, although sometimes third and fourth become necessary.

EXAMPLES.

1. Finding from the tables the logarithms of 232, 233, 234, 235, to be 2.365488, 2.367356, 2.369216, and 2.371068, required to interpolate the logarithm of 233.4.

SOLUTION.

ARGUMENTS.*	FUNCTIONS.	1ST DIFF'S.	2D DIFF'S.	3D DIFF'S.
232	2.365488			
		.001868		
233	2.367356		−.000008	
		.001860		.000000
234	2.369216		−.000008	
		.001852		
235	2.371068			

* In such a case the number is called the *Argument*, and its logarithm the function. This means simply that the logarithm is a function of the number (or argument).

In this case $a = 2.365488$, $D_1 = .001868$, $D_2 = -.000008$, $D_3 = 0$, and $\frac{p}{q} = \frac{7}{5}$. Hence we have

$$\log 232 = a = 2.365488$$

$$\frac{p}{q} D_1 = \frac{7}{5}(.001868) = .002615$$

$$\frac{1}{2} \cdot \frac{p}{q} \cdot \left(\frac{p}{q} - 1\right) D_2 = -\frac{14}{50}(.000008) = -.000002$$

$$\therefore \log 233.4 = 2.368101,$$

which is exactly as it is in the tables.

2. Finding from the table the logarithms of 61, 62, etc., interpolate the logarithm of 62.23.

326. Sch. 2.—When second differences only are to be used, and four functions of the series are known, a convenient and excellent formula is obtained thus: Let the four functions be a, b, c, d, and let it be required to interpolate between b and c. Let $\frac{p'}{q'}$ be the interval from b to the place of the term to be interpolated. Now if we compute from b, instead of from a, the preceding formula will become

$$\text{The interpolated function} = b + \frac{p'}{q'}\left\{ D_1 + \frac{1}{2}\left(\frac{p'}{q'} - 1\right) D_2 \right\},$$

in which D_1 is the second of the first differences, *i. e.*, the one which falls between b and c; or, in general, if we tabulate the differences as above, it is the first difference which falls in the same horizontal line with the function to be interpolated. Again, as the second differences are supposed to be different, it is best to take the arithmetical mean of the two, which mean will also fall in the same horizontal line with the interpolated function.

3. Find by (***326***) the logarithm of 68.53 from the logarithms of 67, 68, 69, 70. (See table.)

ARGUMENTS.	FUNCTIONS.	1ST DIFF'S.	2D DIFF'S.	MEAN OF 2D DIFF'S.
67	1.826075			
		.006434		
68	1.832509		−.000090	
* - - - - -	- - - - - - -	.006340	- - - - - - -	−.0000905
69	1.838849		−.000091	
		.006249		
70	1.845098			

Here we have $b = \log 68 = 1.832509$, $\frac{p'}{q'} = \frac{53}{100}$, $D_1 = .006340$, and $D_2 = -.0000905$. The student should make the substitutions and compare with the table.

327. SCH. 3. — But it is not for interpolating logarithms that this method is chiefly used. For this purpose the method given in (***196***) is preferable. The student will readily discover that the method of (***196***) is identical with that just given if only first differences are used. When great accuracy is required, and the tables used give the logarithms to 8 or 10 places, it sometimes becomes necessary to use mean second differences, as above. It is, however, in Astronomy that Interpolation has its most important applications. Thus, suppose the Right Ascension (analogous to terrestrial longitude) of a planet has been *observed four times* at intervals of, say one day. By interpolation we may find its Right Ascension at each intermediate hour, or point of time. In this problem the Right Ascension is the *function*, and the *time* is the *argument*.

4. The Right Ascension of Jupiter to-day, July 1st, at noon, is 10h. 5m. 38.6s.; July 2d, at noon, it will be 10h. 6m. 18.86s.; on July 3d, 10h. 6m. 59.41s., and July 4th, 10h. 7m. 40.24s. What will it be July 2d, at midnight?

SOLUTION.

ARGUMENTS.*	FUNCTIONS.*	1ST DIFF'S.	2D DIFF'S.	MEAN 2D DIFF'S.
July 1.	10 h. 5 m. 38.6 s.			
		40.26 s.		
July 2.	10 h. 6 m. 18.86 s.		0.29 s.	
		40.55 s.		0.285s.
July 3.	10 h. 6 m. 59.41 s.		0.28 s.	
		40.83 s.		
July 4.	10 h. 7 m. 40.24 s.			

In this case $\frac{p'}{q'} = \frac{1}{2}$, $b = $ 10h. 6m. 18.86s., $D_1 =$ 40.55s., and $D_2 =$ 0.285s. The answer is 10h. 6m. 39.1s.

5. To-day, July 1st, at noon, the moon's declination (distance from the celestial equator) is 6° 38′ 10″.8 north; at 4 o'clock it will be 5° 45′ 51″.3; at 8 o'clock, 4° 53′ 7″.8; at midnight, 4° 0′ 2″.8; and at 4 o'clock in the morning it will be 3° 6′ 38″.7. Interpolate for the intermediate hours.

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* In this example the argument is the *time*, and the function is the Right Ascension, *i. e.*, the Right Ascension is a function of the time.

SECTION II.

PERMUTATIONS.

328. *Combinations* are the different groups which can be made of m things taken n in a group, n being less than m.

ILL.—Taking the 5 letters a, b, c, d, e, we have the 10 following *combinations* when the letters are taken 3 in a group, or, as it is usually expressed, taken 3 and 3: *abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde.* Taken 2 and 2, we have the following 10 combinations: *ab, ac, ad, ae, bc, bd, be, cd, ce, de. It is to be noticed that no two combinations contain the same letters; i. e.*, they are *different* groups.

329. *Permutations* are the *different orders* in which things can succeed each other.

ILL.—Thus the two letters a, b have the two permutations *ab, ba.* The three letters a, b, c have the 6 permutations *abc, acb, cab, bac, bca, cba.*

330. *Arrangements* are permutations of combinations.

ILL.—Taking the 10 combinations of 5 letters taken 3 and 3, and permuting each combination, we get the *arrangements* of 5 letters taken 3 and 3. Thus the combination *abc* gives the 6 arrangements *abc, acb, cab, bac, bca, cba.* In like manner each of the 10 combinations of 5 letters taken 3 and 3 will give 6 arrangements; whence, in all, 5 letters taken 3 and 3 have 60 arrangements.

331. *Prop.*—*The number of Arrangements of* m *things taken* n *and* n *is*

$$m(m-1)(m-2)(m-3) - - - - (m-n+1).$$

DEM.—Let us consider the number of arrangements which can be made of the m letters a, b, c, d, etc., taken 2 and 2. Letting a stand first, we can have *ab, ac, ad*, etc., to $m-1$ arrangements. Letting b stand first, we can have *ba, bc, bd*, etc., to $m-1$ arrangements. Thus taking each of the m letters in turn we can have $m-1$ arrangements in each case, or $m(m-1)$ arrangements in all.

Again, *each* of these $m(m-1)$ 2 and 2 arrangements will give $m-2$ arrangements 3 and 3, by placing before it each of the letters not involved in it. Thus we have $m(m-1)(m-2)$ arrangements of m letters taken 3 and 3.

Once more, *each* of these $m(m-1)(m-2)$ 3 and 3 arrangements will give $m-3$ arrangements 4 and 4, by placing before it each of the letters not involved in it. Thus we have $m(m-1)(m-2)(m-3)$ arrangements of m letters taken 4 and 4.

Finally, we observe the law; *i. e.*, the number of arrangements is equal to the continued product of $m(m-1)(m-2)(m-3)$ - - - - $\{m-(n-1)\}$ or $m(m-1)(m-2)(m-3)$ - - - - $(m-n+1)$.

332. Cor. 1.—*The number of Permutations of* m *things is*

$$1 \cdot 2 \cdot 3 \cdot 4 \text{ - - - - } m.$$

This is evident since arrangements become permutations when the number in a group is equal to the whole number considered; *i. e.*, when $n=m$.

333. Cor. 2.—*If* p *of the* m *letters are alike* (*as each* a), q *others alike,* r *others alike, etc., the number of permutations is*

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \text{ - - - - } m}{\underline{|p} \times \underline{|q} \times \underline{|r} \times \text{etc.}}$$

Thus consider the permutations of *a, b, c, d,* viz., *abcd, bacd, acdb, bcda, acbd, bcad, abdc, badc, adcb, bdca,* etc. Suppose *b* to become *a*, then since for any particular position of *c* and *d*, as in *abcd*, there are as many permutations of the four letters as there can be permutations of the two letters *a* and *b*, viz., 1×2; if *b* becomes *a* there will be 1×2 fewer permutations when these two letters are alike than when they are different, *i. e.*, $\frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 2}$.

So, in general, if p of the letters are alike, there will be $1 \cdot 2 \cdot 3$ - - - - p, or $\underline{|p}$ fewer permutations than if they are all different, etc.

334. Cor. 3.—*The number of Combinations of* m *things taken* n *and* n *is*

$$\frac{m(m-1)(m-2)(m-3) \text{ - - - - } (m-n+1)}{1 \cdot 2 \cdot 3 \cdot 4 \text{ - - - - } n}.$$

Since arrangements are permutations of combinations, the number of arrangements of m things taken n and n is equal to the number of combinations of m things taken n and n multiplied by the number of permutations of n things. Hence the number of combinations is equal to the number of arrangements of m things taken n and n divided by the number of permutations of n things.

Examples.

1. How many permutations can be made of the letters in the word *marble*? Of those in *home*? Of those in *logarithms*?

2. How many arrangements can be made of 10 colors taken 3 and 3? Of 7 colors taken 2 and 2? Taken 3 and 3? 4 and 4? 5 and 5? 6 and 6? 7 and 7? How many *mixtures* in each case, irrespective of proportions?

3. How many different products can be made from the 9 digits taken 2 and 2? 3 and 3? 4 and 4? 5 and 5? 6 and 6? 7 and 7? 8 and 8? 9 and 9?

4. How many different numbers can be represented by the 9 digits taken 2 and 2? 3 and 3? 4 and 4? etc.

5. In a certain district 3 representatives are to be elected, and there are 6 candidates. In how many different ways may a ticket be made up?

6. There are 7 chemical elements which will unite with each other. How many ternary compounds can be made from them? How many binary?

7. How many different sums of money can be paid with 1 cent, 1 3-cent piece, 1 5-cent piece, 1 dime, 1 15-cent piece, 1 25-cent piece, and 1 50-cent piece?

Sug.—If taken 1 and 1, how many? If 2 and 2, how many? If 3 and 3, etc.? How many in all?

8. In how many ways can 12 ladies and 12 gentlemen arrange themselves in couples?

9. If you are to select 7 articles out of 12, how many different choices have you?

10. How many different sums can be made from 1, 2, 3, 4, 5, 6, taken 2 and 2?

11. How many permutations can be made from the letters in the word *p o s s e s s i o n s?* (See ***333.***) How many from the letters in the word *c o n s i s t e n c i e s?*

12. How many different signals can be made with 10 different-colored flags, by displaying them 1 at a time, 2 at a time, 3 at a time, etc., the relative positions of the flags with reference to each other not being taken into account?

Probabilities.

335. The Mathematical Probability of an event is the number of favorable opportunities divided by the whole number of opportunities. *The Mathematical Improbability* is the number of unfavorable opportunities divided by the whole number of opportunities.

Ill.—A man draws a ball from a bag containing 5 white and 2 black balls; the opportunities favorable to drawing a white ball are *five*, and the whole number of opportunities is seven; hence the mathematical probability of drawing a white ball is $\frac{5}{7}$. The mathematical improbability of drawing a white ball is $\frac{2}{7}$.

EXAMPLES.

1. I learn that from a vessel on which my friend had taken passage, one *person* has been lost overboard. There were 40 passengers, and 20 in the crew. What is the probability that my friend is safe? What the improbability? If I learn that a *passenger* is lost, what then is the probability that my friend is safe? What that he is lost?

2. A man fires into a flock of birds of which 6 are white, 4 black, 5 slate-colored, and 3 piebald. If he kills one, what is the probability of its being a black bird? What the improbability of its being piebald? How much more probable is it that he will kill a white than a piebald bird? A black than a piebald?

3. Twenty-three persons sit around a table. What is the probability of any given couple sitting together?

ILL.—Call the two persons A and B. Then wherever A may sit, there are 22 others who *may* sit beside him in one of two places (on his right or left). There are therefore 2 favorable and 20 unfavorable opportunities.

4. What are the odds against the fourth of July coming on Sunday in any year taken at random?

SUG.—The *odds against* an event is the ratio of the unfavorable to the favorable opportunities.

5. The moon changes about once in 7 days. What is the probability that a change of weather will come within 3 days of a change in the moon?

6. The letters *a, e, m, n,* can be arranged so as to form four words, viz., mane, mean, name, amen. If they are arranged at random, what is the probability of their forming a word? What the "odds against" their forming a word?

7. Show that the probability that a leap-year will contain 53 Sundays is $\frac{2}{7}$.

8. Three balls are to be drawn from an urn which contains 5 black, 3 red, and 2 white balls. What is the probability of drawing 2 black balls and 1 red?

SUG'S.—The first question is, How many opportunities in all? That is, how many different groups (*combinations*) can be made of 10 balls taken 3 and 3. Second, How many opportunities favorable to drawing two black balls and one red at the same time? There are 5 black balls, and these can be combined 2 and 2 in $\frac{5 \cdot 4}{1 \cdot 2}$, or 10, ways; and as one of the three red balls can be obtained in 3 ways, each one of these combined with one of the 10 ways of obtaining the black balls will give 10×3, or 30, favorable opportunities for selecting the balls as desired. The probability is therefore $\frac{30}{120}$, or $\frac{1}{4}$.

9. If from a lottery of 30 tickets, marked 1, 2, 3, etc., 4 tickets are drawn, what is the probability that 3 and 5 are among them? What are the odds against it?

Sug's.—From 30 how many combinations of 4 and 4? From 28 how many combinations of 2 and 2? Odds against drawing 3 and 5, 143 to 2.

10. A bag contains a \$5 bill, \$10 bill, and 6 blanks. What is the *expectation* of one drawing? That is, what is one drawing worth?

Sug.—The probability that one draught will take the \$5 bill is $\frac{1}{8}$, and hence is worth $\$\frac{5}{8}$. The probability that the \$10 note will be drawn is also $\frac{1}{8}$, and hence this *expectation* is $\$\frac{10}{8}$. The entire expectation is therefore $\$\frac{15}{8}$, or $\$1.87\frac{1}{2}$. Hence a gambler who should sell such chances at \$2 each, would in the long run make money.

11. What is the expectation of a draught from a bag containing 5 \$2 bills, 4 \$5 bills, 2 \$10 bills, 1 \$100 bill, and 50 blanks?

12. In a given bag are 5 \$2 bills, 3 \$5 bills, and 6 blanks. What is the expectation of 2 draughts?

Sug's.—There are $\frac{14 \cdot 13}{1 \cdot 2} = 91$ opportunities, or ways in which 2 things can be drawn from 14.

There are $\frac{5 \cdot 4}{1 \cdot 2}$ ways in which \$2 bills *may* be drawn. Hence the probability of drawing 2 \$2 bills is $\frac{10}{91}$, and this expectancy is $\$\frac{40}{91}$.

In like manner the probability of drawing 2 \$5 bills is $\frac{3}{91}$, and this expectancy is $\$\frac{30}{91}$.

The probability of drawing 2 blanks is $\frac{15}{91}$, and this expectancy 0.

The probability of drawing 1 \$2 and 1 \$5 bill is $\frac{15}{91}$, and this expectancy is $\$\frac{105}{91}$.

The probability of drawing 1 \$2 bill and 1 blank is $\frac{30}{91}$, and this expectancy is $\$\frac{60}{91}$.

The probability of drawing 1 \$5 bill and 1 blank is $\frac{18}{91}$, and this expectancy is $\$\frac{90}{91}$.

The entire expectancy, or worth, of 2 draughts is therefore $\frac{40}{91} + \frac{30}{91} + \frac{105}{91} + \frac{60}{91} + \frac{90}{91}$ dollars, or $\$3.57\frac{1}{7}$.

Observe that the sum of all the probabilities, *i. e.*, $\frac{10}{91} + \frac{3}{91} + \frac{15}{91} + \frac{15}{91} + \frac{30}{91} + \frac{18}{91}$, is 1, as it should be.

That the probability of drawing 1 \$2 bill and 1 \$5 is $\frac{15}{91}$, is seen thus: There are 5 opportunities favorable to drawing 1 \$2 bill, and with *each* of these there are 3 opportunities favorable to drawing 1 \$5 bill.

13. There are 4 white balls and 3 black ones in one bag, and 2 white ones and 7 black ones in another bag. What is the probability of drawing a white ball from each bag at the first draught from each?

SOLUTION.—There are in all 7 opportunities of drawing a ball from the first bag, and with *each* one of these there are 9 opportunities from the second bag; hence there are 7×9, or 63 opportunities in all. Again, there are 4 favorable opportunities for drawing a white ball from the first bag, and with *each* of these there are 2 favorable opportunities for drawing a white ball from the second bag; *i. e.*, there are in all 4×2, or 8, favorable opportunities. Hence the probability is $\frac{8}{63}$. *Notice that this compound probability is the product of the two simple probabilities.*

14. The probability that A can solve a problem is $\frac{3}{5}$, and that B can do the same is $\frac{2}{7}$, what is the joint probability?

SUG'S.—The student will observe that there are 4 possible results, viz.: 1. *Both* may succeed, of which the probability is $\frac{6}{35}$; 2. A may succeed and B fail, of which the probability is $\frac{15}{35}$; 3. B may succeed and A fail, of which the probability is $\frac{4}{35}$; and 4. Both may fail, of which the probability is $\frac{10}{35}$. Now *either* the first, second, or the third result will give a solution. Hence the probability of success is $\frac{6}{35} + \frac{15}{35} + \frac{4}{35} = \frac{25}{35}$, or $\frac{5}{7}$.

This result may be more expeditiously obtained by considering that they will succeed if both do not fail. The probability of A's failure is $\frac{2}{5}$, and of B's $\frac{5}{7}$. Hence the probability that both will fail is $\frac{5}{7} \times \frac{2}{5}$, or $\frac{2}{7}$; and the probability of success is $1 - \frac{2}{7}$, or $\frac{5}{7}$.

15. It may be said that on an average 10 persons will die during the next 10 years

Out of	every	62	whose	present age is	30,
"	"	45	"	"	40,
"	"	35	"	"	50,
"	"	25	"	"	60.

What is the probability that a person who is 30 will live till he is 60? What that a person who is 40 will live till he is 70?

SUG'S.—Let us examine the probability that the man who is 30 will *die before he is* 60. The probability that he dies before 40 is $\frac{10}{62}$, and that he lives to 40 $\frac{52}{62}$. Now the probability that a man who is 40 dies before 50 is $\frac{10}{45}$. Hence the probability is $\frac{10}{45}$ of $\frac{52}{62}$ that this man lives to 40 and dies between 40 and 50, or it is $\frac{35}{45}$ of $\frac{52}{62}$ that he lives to 50. Finally, the probability that he dies between

50 and 60 is $\frac{10}{35}$ of $\frac{35}{45}$ of $\frac{52}{62}$, or it is $\frac{25}{35}$ of $\frac{35}{45}$ of $\frac{52}{62}$ that he lives from 50 to 60. Hence the probability that a man who is 30 will die before he is 60 is

$$\tfrac{10}{62} + \tfrac{52}{62} \times \tfrac{10}{45} + \tfrac{52}{62} \times \tfrac{35}{45} \times \tfrac{10}{35}, \text{ or } \tfrac{149}{279};$$

and, consequently, the probability that he will live is $1 - \frac{149}{279}$, or $\frac{130}{279}$; *i. e.*, it is a little more probable that a man who is 30 will die before he is 60, than that he will live to 60.

16. What is the probability that two persons, *A* and *B*, aged respectively 30 and 40, will be alive 10 years hence?

SUG'S.—Chance of *A's* being alive $\frac{52}{62}$, of *B's* $\frac{35}{45}$, of both $\frac{52}{62} \times \frac{35}{45}$, or $\frac{182}{279}$.

LOGARITHMS OF NUMBERS.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806180	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

Remark.—In the following Table, the *first two figures*, in the first column of Logarithms, are to be prefixed to each of the numbers, in the same horizontal line, in the next nine columns; but when a point (•) occurs, a 0 is to be put in its place, and the *two initial figures* are to be taken from the next line below.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	•300	•724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	•361	•775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	•195	•600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	•207	•602	•998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	0105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	•380	•766	1153	1538	1924	2309	2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	•320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	••38	•407	•776	1145	1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	9543	9904	•266	•626	•987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	•258	•611	•963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	••26	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	•253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	•245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	••12	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	•194	•508	•822	1136	1450	1763	2076	2389	2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	•142	•449	•756	1063	1370	1676	1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	•168	•469	•769	1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	8977	9264	9552	9839	•126	•413	•699	•985	1272	1558	287
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	••51	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	•029	•303	•577	•850	1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
160	204120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	••51	•319	•586	•853	1121	1388	1654	1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	•193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	••50	•300	250
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	•176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	236
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	•213	•446	•679	•912	1144	1377	1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	8982	9211	9439	9667	9895	•123	•351	•578	•806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	•161	•378	•595	•813	218
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	••56	•268	•481	•693	•906	1118	1330	1542	212
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	322219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	••08	•211	203
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	••47	•245	199
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
220	342423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	••54	194
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	••25	•215	•404	•593	•783	•972	1161	1350	1539	189
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	•143	•328	•513	•698	•883	185
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	••30	181
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	••51	•228	•405	•582	•759	177
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	9674	9847	••20	•192	•365	•538	•711	•883	1056	1228	173
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	•102	•271	•440	•609	•777	•946	1114	1283	1451	169
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	414973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
263	9956	•121	•286	•451	•616	•781	•945	1110	1275	1439	165
264	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
265	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	9752	9914	••75	•236	•398	•559	•720	•881	1042	1203	161
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	•122	•279	•437	•594	•752	158
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
N.	0	1	2	3	4	5	6	7	8	9	D.

ANSWERS.

PART I.

[NOTE.—The full-faced figures in connection with the number of the page refer to the Articles in the text. The numbers in parenthesis in the paragraph refer to the particular Example. * * * indicate that it is not thought expedient to give the answer.]

ADDITION.

(PAGE 13, ***68.***)

(1.) $-7a$. **(2.)** $4a^2-b^2+a^2b+ab^2+2b^3-3a^3$. **(3.)** $15ca^2x^2+2ba^2x^2+7mx^2y^2$. **(4.)** $4x^{\frac{1}{2}}-x^{\frac{1}{3}}+5x^2+5x^2y-ab-x^3-3$. **(5.)** x. **(6.)** $5cz-x^{\frac{1}{2}}+2z^{\frac{1}{2}}$. **(7.)** $(a+c)z+(m-5a)y$ **(8.)** $(2a+2b+3c-2d+e-2n)x^{\frac{1}{2}}+(12a+4n+2)y^{\frac{1}{2}}$. **(9.)** $(a+b+1)x^2+(b-a+1)xy+(a-b+1)y^2$. **(10.)** $(a+m)(x+y)+(b-n)(x-y)$. **(11.)** $3(m+n-2)\sqrt{x-y}$. **(12.)** $ax^{-\frac{1}{2}}+(3-m)y^{-1}+3c$. **(13.)** $\frac{5}{6}\sqrt[5]{a^2-x^2}$. **(14.)** 0. **(15.)** $\frac{7}{6}x^{\frac{2}{3}}+\frac{1}{6}y^{-\frac{1}{3}}-\frac{1}{6}z^{-2}$. **(16.)** $(a+b+c)\sqrt{x^2-y^2}$. **(17.)** $2(a-2m)x^{\frac{1}{2}}+3(m-1)y^2+8cz$.

SUBTRACTION.

(PAGE 15, ***73.***)

(2.) x^3+x^2-2x-8. **(3.)** $-2(x^2+ax)$. **(4.)** $6x^{\frac{1}{2}}+2x^{\frac{3}{2}}$. **(5.)** $4x^{\frac{2}{5}}y^{\frac{2}{5}}$. **(6.)** $10\sqrt[3]{1+x^2}-6ay^{\frac{1}{2}}$. **(7.)** $a(y^2-y)+(10-x)\sqrt{ab}$. **(8.)** $bx(x-b)-4\sqrt{mn}+2$. **(9.)** $2\sqrt{a-b}-\sqrt{ab}$. **(10).** $a-b+c+d$; $5a$; $a-b+c-d$; and $3a+9b$. **(11.)** * * *.

MULTIPLICATION.

(PAGE 20, ***87.***)

(1.) $72a^3bx^5y^6$. (2.) $24m^{s+1}x^{m+n+2-r}$. (3.) $100x^{\frac{11}{6}}y^{\frac{3}{2}}$; and $-9a^{\frac{7}{6}}b^{\frac{7}{6}}$
(4.) $m^{\frac{1}{6}}$; 1; a^6; $m^{\frac{n-m}{mn}}$; $a^{\frac{m+n}{mn}}$; $c^{\frac{25}{12}}$. (5.) $3a^2+10ab-8b^2$. (6.) $x^4+x^2y^2+y^4$.
(7.) $m^6-3m^2n^2o^2+n^6+o^6$. (8.) $a^{2m}-a^{m+n}+a^{m+2}-a^{m+1}+a^{n+1}-a^3$. (9.) $z^4-(a+b+c+d)z^3+(ab+ac+bc+ad+bd+cd)z^2-(abc+abd+acd+bcd)z+abcd$.
(10.) x^6-y^6. (11.) $a^{\frac{3m}{2n}}b^{-\frac{3t}{2s}}+1$. (12.) $20ab^2+30a^{g-f+n}b^{m+n+1}-10a^{f-g+p-q}b^{1-n-m}-15a^{n+p-q-1}$. (13.) * * *. (14.) * * *
(15.) $-a^4+2a^2b^2-b^4+2(a^2+b^2)c^2-c^4$.

(PAGE 21, ***88.***)

(3.) $6a^4-10a^3x-22a^2x^2+46ax^3-20x^4$. (4.) $4a^5-16a^3b^2+10a^2b^3+15ab^4-25b^5$. (5.) a^4-x^4. (6.) $x^5-5x^4+10x^3-10x^2+5x-1$.

DIVISION.

(PAGE 24, ***105.***)

(1.) $m^{\frac{5}{12}}$; $n^{\frac{m+n}{n}}$; $(ab)^{\frac{2mn-1}{n}}$; $\frac{1}{a^2}$; $\frac{1}{a^5}$; $x^{\frac{5}{3}}$; $\frac{1}{x^{\frac{5}{3}}}$. (2.) $\frac{b^2x^2}{a^2y^3}$; $\frac{2ny}{3a^{\frac{1}{2}}m^2x^3}$; $\frac{5a^2bcy}{8dx^5z}$.
(3.) * * *. (4.) Last two, $b^{1-2n}-b^{1-n}-b-b^{1+n}$; $x^{\frac{m}{n}-n-1}-2x^{-\frac{n}{m}-s-2}+3x^{-s}$.
(5 to 11.) * * *. (12.) $(x+y)^3$. (13.) * * *. (14.) * * *. (15.) $a+b$.
(16.) $a^{3m-n}b^{3p+1}c-a^{2m+2n-1}b^2c^n+b^pc^m$. (17.) m^m+an^{an}. (18.) mx^3+nx^2+nx+m. (19.) hx^2-2x+k. (20.) $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}-x^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}+z^{\frac{2}{3}}$.

(PAGE 26, ***106.***)

(2.) $x^2-5ax+4a^2$. (3.) $2a^3+4a^2+8a+16$. (4.) $3y^2-4x^2$. (5.) $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$; x^2+y^2.

(PAGE 27, ***107.***)

(3.) $2y^4-8y^3+12y^2-8y+2$. (4.) $x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5+y^6$; $1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8$ etc. (5.) * * *. (6.) * * *.
(7.) * * *.

FACTORING.

(PAGE 31, ***122.***

EXAMPLES in factoring are, in general, of such a nature that the answers cannot be given without destroying the utility of the problem; hence only the following are given: (**23.**) $k^{\frac{6}{7}}y^2-k^{\frac{5}{7}}m^{\frac{1}{7}}y^{\frac{5}{3}}z^{\frac{1}{5}}+k^{\frac{4}{7}}m^{\frac{2}{7}}y^{\frac{4}{3}}z^{\frac{2}{5}}-k^{\frac{3}{7}}m^{\frac{3}{7}}yz^{\frac{3}{5}}+k^{\frac{2}{7}}m^{\frac{4}{7}}y^{\frac{2}{3}}z^{\frac{4}{5}}-k^{\frac{1}{7}}m^{\frac{5}{7}}y^{\frac{1}{3}}z+m^{\frac{6}{7}}z^{\frac{6}{5}}$. (**24.**) $x^{\frac{3}{7}}-x^{\frac{5}{14}}y^{\frac{1}{14}}+x^{\frac{2}{7}}y^{\frac{1}{7}}-x^{\frac{3}{14}}y^{\frac{3}{14}}+x^{\frac{1}{7}}y^{\frac{2}{7}}-x^{\frac{1}{14}}y^{\frac{5}{14}}+y^{\frac{3}{7}}$. (**26.**) * * *; $1-a=(1+\sqrt{a})(1-\sqrt{a})$; $1+a$ is divisible by $1+\sqrt[3]{a}$, $1+\sqrt[5]{a}$, etc. (**27.**) $10a\left(\frac{x^2}{y}-\frac{y}{x^2}\right)^2$.

GREATEST OR HIGHEST COMMON DIVISOR.

(PAGE 34, ***124.***)

(**1.**) 12. (**2.**) 12. (**3.**) 3. (**5.**) $2k^3l^5m^2$. (**6.**) $2a^2b$. (**7.**) $x^{\frac{2}{3}}y^{-\frac{2}{3}}z^n$. (**8.**) x^2y. (**10.**) $4b^2x-4b^2$.

(PAGE 38, ***129.***)

(**3.**) $x-1$. (**4.**) $x+1$. (**5.**) $2a+3x$. (**6.**) $2a-b$. (**7.**) $4(x^2-2xy+y^2)$. (**8.**) $2ax^3-6ax^2+10ax-2a$.

(PAGE 38, ***130.***)

(**1.**) $x+6$. (**2.**) $2(x+y)$.

LOWEST OR LEAST COMMON MULTIPLE.

(PAGE 39, ***132.***)

(**2.**) $(a+b)^3(a-b)^2$. (**3.**) x^4-4. (**4.**) $4(a^4-2a^2+1)$. (**5.**) $4959a^5b^2x^3y^3$. (**6.**) $1-18a+81a^2$. (**8.**) $(x^3-39x+70)(x-10)$. (**9.**) $(x-1)(x+2)(x-3)$. (**10.**) $(a^3-4a^2b+9ab^2-10b^3)(a+4b)$. (**11.**) $x^4-14x^3+71x^2-154x+120$. (**12.**) $x^5+7x^4-10x^3-70x^2+9x+63$.

FRACTIONS.

(PAGE 48, **167.**)

(1.) * * *. **(2.)** * * *. **(3.)** $1+x+x^2+x^3+$ etc.; $x^2+10-\frac{13}{x^2+4}$; $m+n-\frac{x+y}{m+n}$; $1-a+a^2-a^3+a^4-a^5+a^6-a^7+a^8-a^9$; $1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\frac{x^4}{16}$ + etc.; $a+x$; $x^n+1+x^{-n}+x^{-2n}+x^{-3n}+$ etc.; $1-na^{-2n}+n^2a^{-4n}-n^3a^{-6n}+n^4a^{-8n}-n^5a^{-10n}+$ etc. **(4.)** $3\cdot 7^{-1}a^{-1}y$; $(m+n)^{m+3}$; $c^5d^{-n}z^{2a}$; $x^{-3}(a+b)^3(a-b)^{-3}$. **(5.)** $\frac{1}{1-x}$; $\frac{5}{2}$; $\frac{10x^2+x+4}{5x}$; $\frac{a(a+b)}{a-b}$; $\frac{(b+c)^2-a^2}{2bc}$; $\frac{x(1-x)}{1+x}$. **(6.)** * * *. **(7.)** * * *. **(8.)** $\frac{(a+x)\ (a-x)^2}{x(a^2-x^2)}$, $\frac{x(a-x)^2}{x(a^2-x^2)}$, $\frac{x(a+x)}{x(a^2-x^2)}$. **(9.)** $\frac{(x-y)^2}{(x-y)^3}$, $\frac{x(x-y)}{(x-y)^3}$, $\frac{x^2}{(x-y)^3}$. **(10.)** $\frac{(1+x^3)(1-x^3)}{(1+x)^3(1-x^3)}$, $\frac{(1-x^2)^3}{(1+x)^3(1-x^3)}$. **(11.)** $\frac{m^4(m^4-n^4)}{m^2n^2(m^2+n^2)(m^2-n^2)}$, $\frac{n^4(m^4-n^4)}{m^2n^2(m^2+n^2)(m^2-n^2)}$, etc. **(12.)** $\frac{2(9x^2-16)}{3x(9x^2-16)}$, $\frac{9x^2(3x-4)}{3x(9x^2-16)}$, $\frac{3x(2+3x)}{3x(9x^2-16)}$. **(13.)** $\frac{a+b}{abc}$; $\frac{10a^2}{20b^2y^2+7c^2x^2}$; $\frac{x^2}{9+3xy^2}$; $\frac{adfh-bcfh}{bdeh+bdfg}$; $\frac{m^3+mn^2-m^2n}{m-n}$; $\frac{afd+ae}{bdf+be+cf}$. **(14.)** $\frac{886a}{1155}$; $\frac{6(x-7)}{49}$; $\frac{2}{1-x^4}$; $a-c$; $\frac{5x+1}{x^4-1}$; 0; $\frac{3x^2+2x+13}{x^3+27}$; 0; **(15.)** $\frac{2(a^2+x^2)}{a^3+x^3}$; $a+b+\frac{2abx-3cx^2}{bc}$. **(16.)** $\frac{3b^2-2a^2b-2a-3}{a(b^2-1)}$, $\frac{2}{x^4+x^2+1}$; $\frac{3x^3+20x^2-32x-235}{x^3+8x^2-5x-84}$. **(17.)** $\frac{2a+d+e}{(a+c)(a+d)(a+e)}$; $\frac{y-px-3mpy^2}{(3my^2-x)^2}$; $\frac{x+y}{y}$. **(18.)** 0. **(19.)** $\frac{6x}{x^2-9}$; 4; $\frac{4}{x^2-10x+21}$. **(20.)** $6x+\frac{6a+b}{6}$; $\frac{4ab}{(a-b)^2}$. **(21.)** $\frac{2x-3}{(x^2-1)(2x+3)}$. **(22.)** $\frac{8a}{3}$; $\frac{x^2+b^2}{a}$; 1; $\frac{a^2-ax+ay-xy}{b^2-bx+by-xy}$; $\frac{(a+1)^2}{a}$; $\frac{4}{3}$. **(23.)** $a^{-1}+b$; $-\frac{1}{8}x$; $\frac{4b}{27a^4}$; $1-x^2$; y^{2n}. **(24.)** $\frac{(x^{\frac{1}{3}}-1)^3}{y-1}$; $\frac{a^mc+b^rc-2c^{n+1}}{(a^{2(m-n)}b^{2(n-r)})(2a^m-3b)}$; $l^4-\frac{1}{l^4}$; $\frac{1}{a^2}+\frac{2}{ac}+\frac{1}{c^2}-\frac{1}{b^2}$; $1+\frac{1}{8}x^2-\frac{1}{8}x^3-\frac{1}{64}x^5$; $\frac{x^2-y^2}{2(x^2+y^2)}$; $\frac{a^4+a^2+1}{a^2}$. **(25.)** $\frac{1-y}{x}$. **(26.)** $1+x^2+x^{-2}$; $\frac{4ab}{a^2-b^2}$. **(27.)** $\frac{91a^2}{6b^2}$; $\frac{1}{m^4n^4}$; 1; $\frac{1}{4}$. **(28.)** $\frac{7a^3b^{\frac{4}{5}}}{121m^4y}$; $\frac{1}{1-81a^{18}}$; $\frac{1}{x^{2mn}}$.

(29.) $1+a^2+a^4$; $3(a+b)$; $\frac{x-a}{x+a}$; $\frac{1}{2x^2-1}$; $-\frac{bc(c-b)^2}{c^4+c^2b^2+b^4}$. **(30.)** $m^3-m^2n^{-1}+mn^{-2}-n^{-3}$; $\frac{2(a-b)^2}{3b^2(a+b)}$; $a^2-2ac+c^2-b^2$; 1. **(31.)** 1. **(32.)** 1. **(33.)** $a^{-2}-a^{-1}b^{-1}+b^{-2}-a^{-1}c^{-1}-b^{-1}c^{-1}+c^{-2}$. **(34.)** $\frac{b^5c^4d^7+a^2c^4d^7}{a^2b^5d^7-a^2b^5c^4}$; $\frac{a^{\frac{2}{3}}b^{13}y^n+a^{\frac{2}{3}}b^{13}x^{\frac{2}{3}}}{x^{\frac{2}{3}}y^n}$; $\frac{x^4y^4}{(x-y)^4(x^4+y^4)}$; $\frac{a^3+b^3}{a^2b^2}$. **(35.)** $\frac{cd-1}{c+d}$; $\frac{3}{x\sqrt[3]{5}}$; $\sqrt{\frac{m-n}{m+n}}$; $\frac{(m+n)^c}{(m-n)^c}$. **(37.)** +. **(38.)** −. **(39.)** +. **(40.)** 1; $2a^2-ax-ay$; 1; $(a^3-b^3)^2$.

INVOLUTION.

(Page 59, ***190.***)

(1.) $9a^6$; $4a^{\frac{2}{5}}x^2$; $\frac{9}{25\sqrt{x}}$; $\frac{9}{49}ax^2$; $\frac{25}{4}x$; $\frac{1}{2}$; $\frac{m^4}{x^{\frac{2}{5}}}$. **(2.)** $1-2x+3x^2-2x^3+x^4$; $4a^2-12ax^3+9x^6$. **(3.)** $9-12x-2x^2+4x^3+x^4$; $27x^6-27x^4+9x^2-1$; $1-2x^{\frac{1}{2}}+x$; $x^{\frac{3}{2}}-3xy^{\frac{1}{2}}+3x^{\frac{1}{2}}y-y^{\frac{3}{2}}$. **(4.)** $81a^2x^{\frac{2}{3}}$; $16a^4x^{12}$; $a^{-3m}x^{-m}$; $a^{\frac{3}{2}}x^{\frac{1}{2}}$; $a^2x^{\frac{2}{1}}$; $5^{\frac{2}{3}}x^{\frac{1}{3}}y^{\frac{2}{3}}$; $5^{\frac{m}{n}}x^{\frac{m}{2n}}y^{\frac{m}{n}}$; $\frac{1}{125x^{\frac{3}{2}}y^3}$. **(5.)** $25ax^2$; $512a^9x^{\frac{3}{2}}$; $\frac{(1000)^{\frac{1}{5}}y^{\frac{3}{5}}}{a^{-\frac{3}{10}}}$; $\frac{m^{\frac{2n}{3}}}{(1681)^{\frac{1}{3}}y^{\frac{4}{3}}}$; $\frac{y^{-\frac{3}{14}}}{a^{\frac{1}{7}}x^{-\frac{6}{35}}}$; $\frac{c^{\frac{ps}{qr}}}{a^{\frac{mp}{q}}b^{\frac{mp}{nq}}}$. **(6.)** $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$; $x^4-4x^3y+6x^2y^2-4xy^3+y^4$; $27a^6-27a^4x+9a^2x^2-x^3$; $x^{-5}-5x^{-6}y+15x^{-7}y^2-35x^{-8}y^3+70x^{-9}y^4-$ etc.; $x^{-4}+4x^{-5}y+10x^{-6}y^2+20x^{-7}y^3+35x^{-8}y^4+$ etc.; $5^{\frac{1}{2}}+\frac{x^2}{2\cdot 5^{\frac{1}{2}}}-\frac{x^4}{8\cdot 5^{\frac{3}{2}}}+\frac{x^6}{16\cdot 5^{\frac{5}{2}}}-\frac{5x^8}{128\cdot 5^{\frac{7}{2}}}+$ etc.; $\frac{1}{x}+\frac{a^2}{2x^3}+\frac{3a^4}{8x^5}+\frac{5a^6}{16x^7}+\frac{35a^8}{128x^9}+$ etc.; $1-4x^2+6x^4-4x^6+x^8$; * * *; $\frac{1}{m^3}\left(1+\frac{3x}{m}+\frac{6x^2}{m^2}+\frac{10x^3}{m^3}+\frac{15x^4}{m^4}+\text{etc.}\right)$; * * *; * * *. **(7.)** * * * * *.

EVOLUTION.

(Page 61, ***195.***)

(1.) $2\cdot 2\cdot 2\cdot 59=472$; $2\cdot 73=146$; $5\cdot 7\cdot 67=2345$; * * *; $\pm\{a\cdot c\cdot(a+b)\}=\pm(a^2c+abc)$; * *. **(3 to 6.)** * * * *.

(PAGE 65, ***197.***)

To give the roots in problems in evolution would be to destroy the benefit of the exercise; hence they are omitted.

REDUCTION OF RADICALS.

(PAGE 70, ***20~.***)

(1.) * * * *. (2.) *; $\frac{1}{3}\sqrt[3]{18}$; *; *; *; $\sqrt{7}$; $\frac{1}{a-b}(a^2-b^2)^{\frac{1}{2}}$; * * * * * *; $\frac{x^2+xy+y^2}{5(x+y)}\sqrt{15(x^2-y^2)}$. In such examples be careful to leave only integral forms under the radical sign, in the reduced expression. (3 and 4.) * * * *, $\sqrt{\frac{1}{2}} = \sqrt[4]{\frac{1}{4}}$. (5.) * * *, $a^3\left(1-\frac{b^2}{a^2}\right)^{\frac{3}{2}}=(a^2-b^2)^{\frac{3}{2}}$. (6.) $\sqrt[6]{8}$ and $\sqrt[6]{9}$; * * * *; $\sqrt{(x-y)^2}$ and $\sqrt{x+y}$. (7.) * * *. (8.) $\frac{2a}{3}\sqrt{15}$; * * * *; $\frac{x^{\frac{1}{m}}y^{\frac{n-1}{n}}}{y}$; * * *; $\frac{\sqrt[3]{5}\sqrt[5]{81}}{3}$. (9.) $\frac{1}{x-y}\sqrt{x^3-y^3}$; $\frac{2x}{9-3x^2}\sqrt[3]{(3-x^2)^2}$; $\frac{x^2-x\sqrt{y}}{x^2-y}$; $\frac{x-2\sqrt{xy}+y}{x-y}$; $\sqrt{5}-\sqrt{2}$. $\frac{3(\sqrt{3^5}-9\sqrt[3]{5}+\sqrt{3^3}\sqrt[3]{5^2}-15+\sqrt{3}\sqrt[3]{5^4}-\sqrt[3]{5^5})}{2}$; $2(\sqrt[3]{25}+\sqrt[3]{20}+\sqrt[3]{16})$; $11\sqrt{2}-4\sqrt{15}$; $\frac{3\sqrt{5}+2\sqrt{10}+3\sqrt{3}+2\sqrt{6}}{2}$; $\frac{x^2+x\sqrt{a^2+x^2}}{a^2}$; $2x^2+1-2x\sqrt{x^2+1}$; $-\left(x+\sqrt{x^2-1}\right)$; $x^2+x+\sqrt{x^4+2x^3+x^2-1}$; $2(-\sqrt{6}+\sqrt{2}+2)$; $\frac{2\sqrt{3}+\sqrt{30}-3\sqrt{2}}{6}$. (10.) $x^{\frac{14}{3}}+x^{\frac{13}{3}}y^{\frac{1}{5}}+x^4y^{\frac{2}{5}}+x^{\frac{11}{3}}y^{\frac{3}{5}}+x^{\frac{10}{3}}y^{\frac{4}{5}}+x^3y+x^{\frac{8}{3}}y^{\frac{6}{5}}+x^{\frac{7}{3}}y^{\frac{7}{5}}+x^2y^{\frac{8}{5}}+x^{\frac{5}{3}}y^{\frac{9}{5}}+x^{\frac{4}{3}}y^2+xy^{\frac{11}{5}}+x^{\frac{2}{3}}y^{\frac{12}{5}}+x^{\frac{1}{3}}y^{\frac{13}{5}}+y^{\frac{14}{5}}$; $x^{\frac{22}{3}}+x^{\frac{20}{3}}y^{\frac{3}{4}}+x^6y^{\frac{3}{2}}+x^{\frac{16}{3}}y^{\frac{9}{4}}+x^{\frac{14}{3}}y^3+x^4y^{\frac{15}{4}}+x^{\frac{10}{3}}y^{\frac{9}{2}}+x^{\frac{8}{3}}y^{\frac{21}{4}}+x^2y^6+x^{\frac{4}{3}}y^{\frac{27}{4}}+x^{\frac{2}{3}}y^{\frac{15}{2}}+y^{\frac{33}{4}}$; $(\sqrt{8}+\sqrt{3}-\sqrt{5})(3-2\sqrt{6})$. (11 to 13.) * * *.

COMBINATION OF RADICALS.

(PAGE 74, ***217.***)

(1.) * * *. (2.) * * *; $\frac{2a^2}{x^2}$; $\frac{2a}{x}$. (3, 4, 5.) * * *. (6.) $\sqrt[15]{864}$; $\frac{1}{15}\sqrt[6]{151875}$; $\sqrt[12]{8a^7x^{11}}$; $6x\sqrt[10]{xy^7}$; $\sqrt[6]{1+5x^2+10x^4+10x^6+5x^8+x^{10}}$

$\frac{1}{3}\sqrt[15]{49\times 3^{1}}$; $3\sqrt[6]{32}$; 30; $12\sqrt{3}$. (7.) 41; $x+y$; $246+58\sqrt{30} - 4\sqrt{5} - 36\sqrt{6}$; $3\sqrt[3]{20} - 12\sqrt[3]{3} - \sqrt[3]{180} + 12$. (8.) $\frac{3}{4}\sqrt{10}$; $4\sqrt[3]{9}$; $\sqrt{3\sqrt{2}}$; $\frac{3}{4}\sqrt[4]{40}$. (9.) $\frac{11}{8}+\frac{1}{4}\sqrt{2}$; $2x\sqrt[4]{x}$; $\frac{2}{3}\sqrt{6}+\sqrt{2}$; $b\sqrt{x}$; $\sqrt[4]{\frac{b}{a}}$; $\frac{a+b}{(a-b)(a+1)}\sqrt{a^2-1}$; $\sqrt{a}+\sqrt{b}+\sqrt{c}$; $\frac{x^4+a^2x^2}{a^4}$. (10.) $9x\sqrt[3]{4x}$; $\frac{2ax^3}{243}\sqrt[3]{4a^2x}$; $-\frac{8}{225}\sqrt{15}$; $5-2\sqrt{6}$; $27(a-x)\sqrt{a-x}$; $a^{\frac{3}{2}}-3ab^{\frac{1}{2}}+3a^{\frac{1}{2}}b-b^{\frac{3}{2}}$.
(11.) $9x\sqrt[6]{5y^4}$; $\frac{1}{3}x^2\sqrt[6]{y}$; $b\sqrt{5\sqrt[4]{y}}$; $2\sqrt[5]{7x\sqrt[3]{3x}}$; $\sqrt{1-x}$; $\frac{1}{5}\sqrt[6]{3125x}$; $\frac{1}{3}\sqrt[4]{3}$; (12.) $2+3\sqrt{5}$; $2\sqrt{3}+3\sqrt{5}$; $a\sqrt{x}-\sqrt{ax}$; * * *.

IMAGINARY QUANTITIES.

(PAGE 78, *223.*)

(2.) $12(\sqrt{3}+1)\sqrt{-1}$; $19a\sqrt{-1}$; $(4b^3+3c)\sqrt{-1}$. (4.) $(\sqrt[2n]{x}\pm\sqrt[2n]{y})\sqrt{-1}$.
(5.) $1\sqrt{-1}$; $12(\sqrt{3}-1)\sqrt{-1}$; $11a\sqrt{-1}$; $(a\sqrt{b}-c\sqrt{d})\sqrt{-1}$.
(6.) $1\sqrt{-1}$. (7.) $5\sqrt{2}\sqrt{-1}$; $16\sqrt{3}\sqrt{-1}$; $1\sqrt[4]{-1}$; $\sqrt[4]{a}\sqrt[4]{-1}$.

(PAGE 79, *225.*)

$5-7\sqrt{-1}$, and $9\sqrt{-1}-1$; $2a+(\sqrt{b}+\sqrt{c})\sqrt{-1}$, and $(\sqrt{b}-\sqrt{c})\sqrt{-1}$.

MULTIPLICATION AND INVOLUTION OF IMAGINARIES.

(PAGE 80, *226.*)

(2.) *, *, $-6\sqrt{6}$. (7.) *, 3, $\sqrt{8}$. (9.) $278\sqrt{-3}$, or $278\sqrt{3}\sqrt{-1}$; $972\sqrt{-2}$, or $972\sqrt{2}\sqrt{-1}$. (10.) $xyzw$.

DIVISION OF IMAGINARIES.

(PAGE 81, *227.*)

(3.) $-\frac{1}{4}\sqrt[3]{4}\sqrt{-1}$; $-\sqrt[4]{a}\sqrt{-1}$. (6.) $\frac{a^3-3ab^2}{a^2-b^2}$.

PART II.

SIMPLE EQUATIONS.

(PAGE 87, *28.*)

(1.) 12; 24; $23\frac{1}{4}$; $3\frac{5}{13}$; 8; 4; $\frac{3b}{3a-2b}$; $\frac{b}{c}$; $\frac{a+b+c}{\sqrt{3}}$; 2.9; $2\frac{36}{37}$.

(2.) $b+\frac{bc-b^2}{a}$; 5; $\frac{n(q-p)}{m}$; $\frac{5a(2b-a)}{3c-d}$; $\frac{8ab^2+4b^3-12a^2b}{3a^2+ab-ac+bc}$; $\frac{2\kappa^2}{28p+5g^2}$; $\frac{adfh+bcfh+bdeh+bdfg}{bdfhk}$; 8; 0; 3. (3.) $\frac{8a}{25}$; 4; 4; $4\frac{1}{2}$. (4.) 81; c^2-2bc; 16; 5; $4(a-1)$; 8; 1; 6; 6; 3; $\pm\frac{2a\sqrt{b}}{b+1}$; $\frac{81}{a}$; 4; $\left(\frac{2b+\sqrt{ab}}{a}\right)^2$; $a\left\{1-\left(\frac{2\sqrt{b}}{b+1}\right)^4\right\}$; $\frac{4m^2}{(m+1)^2}$; $\frac{a}{2b}(b-1)^2$; $\frac{9}{7}$; $\sqrt{a^2-\left(\frac{b^3-2a}{3b}\right)^3}$; $\frac{(a-1)^2}{2a}$; $\frac{3}{16}$; $\frac{3a}{4}$.

APPLICATIONS OF SIMPLE EQUATIONS.

(PAGE 90, *33.*)

(1.) *A*'s 84, *B*'s 42, *C*'s 14. (2.) *A*'s $\frac{mns}{1+n+mn}$, *B*'s $\frac{ns}{1+n+mn}$, *C*'s $\frac{s}{1+n+mn}$. (3.) 30, 18. (4.) $\frac{s+d}{2}$, $\frac{s-d}{2}$. (6.) 35. (7.) $\frac{mna}{mn-m-ns}$. (8.) $\frac{5}{8}$. (9.) $\frac{am+bn}{an+bn}$. (13.) $\frac{105}{136}$ of an hour; $\frac{mns}{ns+ms+mn}$. (14.) 317, 951, 1268, 2219; $\frac{s}{3+4m}$, $\frac{ms}{3+4m}$, $\frac{s(1+m)}{3+4m}$, $\frac{s(1+2m)}{3+4m}$. (15.) 90; $\frac{a+b}{p+2-m-n}$. (16.) * * *, $\frac{mpa-pa+mb-na+mc}{m+n-mp+p}$. (17.) 3; $\frac{b^2-ac}{a-2b+c}$. (18.) 19, 30; $\frac{a+b+mc}{m+1}$, $\frac{am-b-mc}{m+1}$. (19.) 73, 77; $\frac{a-mb-mc}{2m}$, $\frac{a-mb+mc}{2m}$. (20.) 3. (21.) $\frac{na}{m+n}$, $\frac{ma}{m+n}$. (24.) 20, 40, 60; $16\frac{2}{3}$, $33\frac{1}{3}$, 50; $14\frac{2}{7}$, $28\frac{4}{7}$, $42\frac{6}{7}$. (26.) 1200. (27.) 50. (28.) 5712. (30.) 50.

SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES.

(PAGE 96, ***42.***)

(1.) $x=10, y=3$. **(2.)** 19, 2.† **(3.)** 16, 35. **(4.)** 7, 2. **(5.)** 7, 17. **(6.)** 2, 2. **(7.)** 12, 9. **(8.)** −2, 19. **(9.)** −2, 1. **(10.)** $\frac{1067}{236}, \frac{278}{59}$. **(11.)** $\frac{1}{ab}, \frac{1}{cd}$. **(12.)** $\frac{2b^2-6a^2+d}{3a}, \frac{3a^2-b^2+d}{3b}$. **(13.)** $\frac{a^2}{b+c}, \frac{b^2-c^2}{a}$. **(14.)** $(a+b)^2, (a-b)^2$. **(15.)** 10, 5. **(16.)** 18, 12. **(17.)** $\frac{1}{a}, \frac{1}{b}$. **(18.)** $\frac{61}{92}, \frac{61}{103}$. **(19.)** $\frac{1}{n}, \frac{1}{m}$. **(20.)** 20, 5; 6, 8; 7, 10; $8y^2-72=0$; $y^2-22y+120=0$; $y^8-4y^6+14y^4-20y^2+9=0$.

APPLICATIONS.

(PAGE 98, ***42.***)

(1.) $18\frac{3}{4}, 31\frac{1}{4}$. **(2.)** 3. **(3.)** 20, 8. **(4.)** 5000, 5000. **(5.)** $\frac{4}{5}$. **(6.)** 24. **(7.)** 29, 32. **(8.)** 5000, 6. **(9.)** $\frac{pm+qn-qmn}{mn-m-n}, \frac{pmn-qn-pm}{mn-m-n}$. **(10.)** $\frac{bm+a}{mn-1}, \frac{an+b}{mn-1}$. **(11.)** 48, 16. **(12.)** 24, 32.

SIMPLE EQUATIONS WITH MORE THAN TWO UNKNOWN QUANTITIES.

(PAGE 101, ***43.***)

(2.) 4, 3, 2. **(3.)** 2, 3, 4. **(4.)** 24, 60, 120. **(5.)** 64, 72, 84. **(6.)** 3, 2, 1. **(7.)** 25, 55, 65. **(8.)** $\frac{b^2+c^2-a^2}{2bc}, \frac{a^2+c^2-b^2}{2ac}, \frac{a^2+b^2-c^2}{2ab}$. **(9.)** $\frac{7}{6}, -\frac{7}{2}, \frac{21}{10}$. **(10.)** $\frac{2}{5}, \frac{2}{3}$, 2. **(11.)** $2a, 2b, 2c$. **(12.)** $\frac{1}{(a-b)(a-c)}, \frac{1}{(b-a)(b-c)}, \frac{1}{(c-a)(c-b)}$. **(13.)** $-\frac{2bc}{b+c}, -\frac{2ac}{a+c}, -\frac{2ab}{a+b}$. **(14.)** 12, 5, 7, 4. **(15.)** 2, 1, 3, −1, −2. **(16.)** 3, 4, 5, 1, 2. **(17.)** $b+c-a, a+c-b, a+b-c$.

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† The values of the unknown quantities are given in the order x, y, z, etc.

APPLICATIONS.

(PAGE 102.)

(2.) \$2, 20 cents, 10 cents. (3.) £3000 at 4%, etc. (4.) $\frac{a}{37}$, $\frac{19a}{37}$, $\frac{25a}{37}$, $\frac{28a}{37}$. (5.) 142857. (6.) 26, 9, 5. (7.) 140, 60, 45, 80. (8.) $18\frac{6}{13}$, $34\frac{2}{7}$, $23\frac{1}{11}$, 80.

RATIO.

(PAGE 105, ***50.***)

(1.) 2; $\frac{1}{2}$; $\frac{3}{4}$; $\frac{5a}{3}$; $x+y$; $\frac{9}{16}$; $\frac{bm}{an}$; $a-b$. (2.) $\frac{1}{2}$; 2; $\frac{4}{3}$; $\frac{3}{5a}$; $\frac{1}{x+y}$; $\frac{16}{9}$; $\frac{an}{bm}$; $\frac{1}{a-b}$. (3.) $5:11$; $1:a^2+b^2$; $2(a-x):(a+x)$. (4.) $9:25$; $a^2:b^2$; $27:125$; $a^3:b^3$; $5:4$; $\sqrt{3}:\sqrt{7}$; $\sqrt{m}:\sqrt{n}$; $3:4$; $\sqrt[3]{x}:\sqrt[3]{y}$. (5.) The former. (7.) $4:1$.

PROPORTION.—APPLICATIONS.

(PAGE 111.)

(3.) 13, 26, 39. (4.) 8, 6. (5.) $\frac{n}{2m}+\frac{b}{2a}$, $\frac{n}{2m}-\frac{b}{2a}$. (6.) 120, 160, 200. (7.) 8:9. (8.) 252. (9.) 56, 84, 70. (10.) 20. (11.) 150. (12.) 300. (14.) 3h. $49\frac{1}{11}$m., 3h. $32\frac{8}{11}$m., 3h. $16\frac{4}{11}$m. (15.) Every $1\frac{1}{11}$ hours, $\frac{6}{11}$ hours, and $1\frac{1}{11}$; or 11 times in 12h., 22 times, and 11 times. (16.) No; since it takes the minute hand $1\frac{1}{11}$ hours to gain a round, and $\frac{6}{11}$ to gain half a round. (17.) 8:45 A.M. (18.) 1st. $\frac{a}{M-m}$, $\frac{a+s}{M-m}$, etc.; 2d. $\frac{s-a}{m-M}$, $\frac{2s-a}{m-M}$, etc.; $\frac{a+mt}{M-m}$, $\frac{s+a+mt}{M-m}$, $\frac{2s+a+mt}{M-m}$, etc.; $\frac{s-a-mt}{m-M}$, $\frac{2s-a-mt}{m-M}$, etc.; $\frac{s-a+Mt}{M-m}$, $\frac{2s-a+Mt}{M-m}$, etc. * * *.

ARITHMETICAL PROGRESSION.

(PAGE 117, ***83.***)

(1.) 83, 903. (2.) -39, -384. (3.) $\frac{3n-1}{6}$, $\frac{n(3n+1)}{12}$. (4.) 0, $\frac{n-1}{2}$.

(5.) 193 · 243 ·· 293 ·· 343 ·· 393 ·· 443. (7.) $-46, \frac{2}{5}$. (8.) 100. (9.) $\frac{3m+n}{4}$, $\frac{m+n}{2}$, $\frac{m+3n}{4}$.

GEOMETRICAL PROGRESSION.

(PAGE 120, **90.**)

(1.) 46875, 58593. (2.) 6, 18, 54, 162, 486. (3.) 16384, $21845\frac{21}{64}$. (4.) $-\frac{1}{96}$, $-\frac{43}{96}$. (5.) $\frac{3}{4}, \frac{9}{8}, \frac{27}{16}, \frac{81}{32}$. (6.) $\frac{27}{8}$; .3; $\frac{49}{90}$; $\frac{46}{99}$. (10.) $-\frac{3}{10}[(-\frac{2}{3})^n-1]$; $\frac{29524}{19683}$; $\frac{3}{2}$; $\frac{132860}{59049}$; $\frac{9}{4}$.

VARIATION.

(PAGE 124, **95.**)

(6.) $x \propto z$. (12.) 18. (13.) $s=\frac{1}{2}ft^2$.

HARMONIC PROPORTION AND PROGRESSION.

(PAGE 126, **100.**)

(6.) $\frac{1}{8}, \frac{1}{11}, \frac{1}{14}$.

PURE QUADRATICS.

(PAGE 128, **108.**)

(1.) ± 4. (2.) ± 5. (3.) $\pm\sqrt{2ab-b^2}$. (4.) $\pm\sqrt{6}$. (5.) $\pm\frac{1}{2}a\sqrt{3}$. (6.) ± 6. (7.) $\pm\sqrt{\frac{b}{a-1}}$. (8.) $\pm\sqrt{-3}$. (9.) $\pm\frac{2a}{5b}\sqrt{5}$. (10.) $\pm 3\sqrt{-n}$. (11.) $\pm\frac{6}{5}\sqrt{-10}$. (12.) $\pm\frac{1}{2}$. (13.) $\frac{\pm 2}{\sqrt{4ab-b^2}}$. (14.) $\pm\frac{a(\sqrt{b}\mp 1)^2}{2\sqrt{b}}$.

APPLICATIONS.

(1.) 12, 20. (2.) $\pm\frac{1}{2}a\sqrt{3}$. (3.) $\frac{\pm m\sqrt{s}}{\sqrt{m^2+n^2+p^2}}$, $\frac{\pm n\sqrt{s}}{\sqrt{m^2+n^2+p^2}}$, $\frac{\pm p\sqrt{s}}{\sqrt{m^2+n^2+p^2}}$. (4.) 4550. (5.) $\frac{n}{\sqrt{2n-m}}$, $\frac{m-n}{\sqrt{2n-m}}$. (6.) 5.57+, 18.12−, 40.51+. (7.) 149,247.2+ miles from the surface of the earth (8.) 240. (9.) $\frac{132\sqrt{7}}{\sqrt{7}\pm\sqrt{17}}$ from A. (10.) $\frac{a\sqrt{3}}{\sqrt{3}\pm 1}$ from the louder bell.

AFFECTED QUADRATICS.

(PAGE 133, ***114.***)

(1.) 8, −2. **(2.)** 6, 2. **(3.)** $a(2 \pm \sqrt{11})$. **(4.)** 8, −1. **(5.)** $2 \pm \sqrt{-41}$. **(6.)** 1, $-a$. **(7.)** 2, −1. **(8.)** 7, $\frac{1}{3}$. **(9.)** $\frac{b^2}{ac}$, $\frac{b^2}{ac}$. **(11.)** 3, $-6\frac{5}{6}$. **(12.)** 5, $-\frac{17}{3}$. **(13.)** $\frac{a \pm b}{ab}$. **(14.)** $\frac{3}{2}$, $-\frac{3}{10}$. **(15.)** $11\frac{523}{625}$, 6. **(16.)** 2, $-\frac{3}{7}$. **(17.)** $\frac{ac}{b}$, $\frac{bc}{a}$. **(18.)** $\frac{1}{6}(-1 \pm \sqrt{133})$. **(19.)** 4, $\frac{1}{9}$. **(20.)** $\frac{1}{8}a(-3 \pm \sqrt{-7})$. **(21.)** $\pm \frac{1}{2}\sqrt{3}$. **(22.)** 8, $-\frac{8}{9}$. **(23.)** $\frac{a^2-3ab+2b^2}{a+b}$, b. **(24.)** 4, $-\frac{5}{3}$. **(25.)** 12, 4. **(26.)** 4, $\frac{1}{4}$. **(27.)** 7.12+, −5.73+. **(28.)** $\frac{1}{2}a(1 \pm 3\sqrt{-3})$. **(29.)** 1, $\frac{1}{81}$. **(30.)** 5, 3.

HIGHER EQUATIONS SOLVED AS QUADRATICS.

(PAGE 136, ***122.***)

(1.) ± 3, $\pm 3\sqrt{-1}$. **(2.)** 2; the other four roots not required. **(3.)** $\pm m^{\frac{3}{2}}$. **(4.)** 27. **(5.)** 121. **(6.)** 64. **(7.)** $b^{\frac{n}{m}}$. **(8.)** ± 8. **(9.)** $\pm\sqrt{-6}$, $\pm\sqrt{2}$. **(10.)** $\sqrt[m]{\frac{1}{2}(-1 \pm \sqrt{4p+1})}$. **(11.)** 4, $\sqrt[3]{49}$. **(12.)** $\left\{\frac{1}{2a}\left(-b \pm \sqrt{4ac+b^2}\right)\right\}^{\frac{1}{3}}$. **(13.)** $\sqrt[n]{(a \pm \sqrt{a^2+b})^2}$. **(14.)** 243, $(-28)^{\frac{5}{3}}$. **(15.)** 16, $\frac{14641}{625}$. **(16.)** $\left\{\frac{1}{2a}\left(b \pm \sqrt{b^2+4ac}\right)\right\}^{3}$. **(17.)** 8, $\frac{125}{64}$. **(18.)** $\sqrt[2n]{8}$, $\sqrt[2n]{-\frac{512}{27}}$. **(19.)** 1, 1, $1 \pm 2\sqrt{15}$. **(20.)** 9, −12. **(21.)** 3, $-\frac{1}{2}$, $\frac{1}{4}(5 \pm \sqrt{1329})$. **(22.)** 4, 69. **(23.)** $\frac{1}{2}(1 \pm \sqrt{5})$. **(24.)** $\frac{a}{2}(1 \pm \sqrt{5})$. **(25.)** 3, $\frac{1}{3}$, $\frac{1}{3}(-8 \pm \sqrt{55})$. **(26.)** 2, $1\frac{1}{2}$, $\frac{1}{4}(7 \pm \sqrt{33})$. **(27.)** $\frac{(1 \pm \sqrt{5})^m - 2^m}{(1 \pm \sqrt{5})^m + 2^m}$. **(28.)** 5, −1, $2 \pm \sqrt{-14}$. **(29.)** 5, −2, $\frac{1}{2}(3 \pm \sqrt{-15})$. **(30.)** 2, 3, 1. **(31.)** 2, $-\frac{3}{2}$, $\frac{1}{4}(1 \pm \sqrt{-43})$. **(32.)** 1, −3, −3. **(33.)** 2, $2 \pm \sqrt{-1}$. **(34.)** 5, $4 \pm \sqrt{7}$. **(35.)** −2, −1, −5. **(36.)** 6, 20, 3. **(37.)** 6, 4, 5. **(38.)** 1, 1, −2, −2. **(39.)** 4, 1, 3, 2. **(40.)** 3, −1, $1 \pm \sqrt{-6}$. **(41.)** 5, −4, 3, −2. **(42.)** 3, −3, $\frac{1}{6}(-13 \pm \sqrt{-155})$. **(43.)** 4, 3, $\frac{1}{2}(7 \pm \sqrt{69})$. **(44.)** 9, 4, $\frac{1}{2}(-3 \pm \sqrt{-7})$. **(45.)** $+1, -1, \pm\sqrt{-1}$; -1, $\frac{1}{2}(1 \pm \sqrt{-3})$; 1, $\frac{1}{2}(-1 \pm \sqrt{-3})$; $1, -1, \pm\sqrt{-1}$; $\frac{\pm 1 \pm \sqrt{-1}}{\sqrt{2}}$; $\pm\sqrt{-1}$, $\frac{1}{2}(\pm\sqrt{3} \pm \sqrt{-1})$;

$\pm 1, \quad \pm\sqrt{-1}, \quad \dfrac{\pm 1 \pm \sqrt{-1}}{\sqrt{2}}; \quad \tfrac{1}{2}\left(\pm\sqrt{2 \pm \sqrt{2}} \pm \sqrt{-2 \pm \sqrt{2}}\right).$

(46.) $\dfrac{2a \pm \sqrt{2(1+a)}}{2(1-a)} \pm \sqrt{\left(\dfrac{2a \pm \sqrt{2(1+a)}}{2(1-a)}\right)^2 - 1}.$ **(48.)** $\tfrac{1}{2}(a \pm \sqrt{a^2-4}).$

(49.) $\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3 \pm \sqrt{8(a+1)}}.$ **(50.)** $\tfrac{1}{2}(1 \pm \sqrt{5}).$ **(51.)** $2, -\tfrac{1}{2}, \tfrac{1}{4}(3 \pm \sqrt{505}).$

(52.) $1, 1, -2, -2.$ **(53.)** $\dfrac{a}{2}(-1 \pm \sqrt{5}).$ **(54.)** $\pm\dfrac{\sqrt{a}}{2}\left\{\dfrac{a}{\sqrt[4]{a^2+2}} + \sqrt[4]{a^2+2}\right\},$

$\pm\dfrac{\sqrt{a}}{2}\left\{\dfrac{a}{\sqrt{a^2+2}} - \sqrt[4]{a^2+2}\right\}\sqrt{-1}.$ **(55.)** $\pm\dfrac{1}{a}\{(\sqrt{1+a^2}-1)(\sqrt{1-a^2}+1)\}^{\frac{1}{2}}, 4, \tfrac{1}{4}.$

(56.) $0, \tfrac{1}{2}, \tfrac{1}{3}. \; -1, +2, -2.$ **(57.)** $\tfrac{1}{2}, \tfrac{1}{4}(-1 \pm \sqrt{-35}), \; \pm 1, \pm\sqrt{\tfrac{1}{6}(-11 \pm \sqrt{85})}.$

SIMULTANEOUS QUADRATICS.

(PAGE 142, ***127.***)

(1.) $x=3, -\tfrac{53}{27}$; $y=-4, \tfrac{227}{27}$. **(2.)** $x=\pm\sqrt{\tfrac{5}{2}}$; $y=2\mp\sqrt{\tfrac{5}{2}}$. **(3.)** $x=2$; $y=2$. **(4.)** $x=\pm 7, \pm 4$; $y=\pm 4, \pm 7$. **(5.)** $x=\pm 3, \pm\tfrac{5}{2}\sqrt{2}$; $y=\pm 2, \pm\tfrac{1}{2}\sqrt{2}$. **(6.)** $x=\pm 2, \pm\tfrac{1}{5}\sqrt{10}$; $y=\pm\tfrac{1}{2}, \mp\tfrac{2}{5}\sqrt{10}$. **(7.)** $x=\pm 3, \mp 8$; $y=\pm 5$. **(8.)** $x=\pm\tfrac{6}{7}\sqrt{14}$; $y=\pm\tfrac{1}{7}\sqrt{14}$. **(9.)** $x=\pm\tfrac{5}{7}\sqrt{21}$; $y=\pm\tfrac{1}{7}\sqrt{21}$. **(10.)** $x=\pm 1, \pm\tfrac{11}{5}\sqrt{-5}$; $y=\pm 2, \pm\tfrac{4}{5}\sqrt{-5}$. **(11.)** $x=\pm 2, \pm\tfrac{4}{3}\sqrt{3}$; $y=\pm 6, \pm\tfrac{10}{3}\sqrt{3}$. **(12.)** $x=\pm 10, \pm\tfrac{14}{47}\sqrt{-47}$; $y=\pm 3, \mp\tfrac{57}{47}\sqrt{-47}$. **(13.)** $x=4, 2, \tfrac{1}{6}(-13 \pm \sqrt{377})$; $y=2, 4, \tfrac{1}{6}(-13 \mp \sqrt{377})$. **(14.)** $x=4, -2, 0$; $y=2, -4, 0$. **(15.)** $x=2, 3$; $y=3, 2$. **(16.)** $x=\pm 3\sqrt{2}$; $y=\pm\sqrt{2}$. **(17.)** $x=9, 4$; $y=4, 9$. **(18.)** $x=11, \tfrac{1}{2}(1 \pm \sqrt{-211})$; $y=3, \tfrac{1}{2}(-15 \pm \sqrt{-211})$. **(19.)** $x=15, 0$; $y=45, 0$. **(20.)** $x=\pm\sqrt{2}$; $y=2\mp\sqrt{2}$. **(21.)** $x=0, 2$; $y=-2, 0$. **(22.)** $x=1, 4$; $y=4, 1$. **(23.)** $x=1, 3, 2\mp 5\sqrt{-1}$; $y=3, 1, 2\pm 5\sqrt{-1}$. **(24.)** $x=5, -2, \tfrac{1}{2}(3 \pm \sqrt{-67})$; $y=2, -5, \tfrac{1}{2}(-3 \pm \sqrt{-67})$. **(25.)** $x=\pm 3, \pm 2$; $y=\pm 2, \pm 3$. **(26.)** $x=\pm 2, \pm 1, \mp 2\sqrt{-1}, \pm\sqrt{-1}$; $y=\pm 1, \pm 2, \pm\sqrt{-1}, \mp 2\sqrt{-1}$. **(27.)** $x=\sqrt[4]{\tfrac{1}{2}(\sqrt{2}-1)}$; $y=\dfrac{1}{\sqrt[4]{2(\sqrt{2}-1)}}$. **(28.)** $x=\dfrac{2abc}{ac+bc-ab}$, $y=\dfrac{2abc}{ab+bc-ac}$, $z=\dfrac{2abc}{ab+ac-bc}$. **(29.)** $x=\pm 3$, $y=\pm 2$, $z=\pm 1$. **(30.)** $x=\pm 2$, $y=\pm 4$, $z=\pm 6$. **(31.)** $x=1$, $y=2$, $z=3$. **(32.)** $x=\pm\tfrac{13}{2}, \mp\tfrac{26}{5}\sqrt{-1}, \pm 5, \mp 4\sqrt{-1}$; $y=\pm\tfrac{26}{5}, \pm\tfrac{13}{2}\sqrt{-1}, \pm 4, \pm 5\sqrt{-1}$. **(33.)** $x=\pm\tfrac{3}{2}\sqrt[4]{2}, \pm 3, \pm 3\sqrt{-1}$; $y=\pm\sqrt[4]{2}, \pm 1, \pm\sqrt{-1}$. **(34.)** $x=8, 0$; $y=8, 0$. **(35.)** $x=2, 8$; $y=8, 2$. **(36.)** $x=10\mp 4\sqrt{6}, 10\mp\tfrac{5}{2}\sqrt{15}$; $y=10\pm 4\sqrt{6}, 10\pm\tfrac{5}{2}\sqrt{15}$. **(37.)** $x=\pm\dfrac{a}{3}\sqrt{\pm 15}, \pm\dfrac{a}{3}\sqrt{\pm 33},$

$\pm a\sqrt{\mp 1}$; $y = \pm\frac{2a}{3}\sqrt{\pm 3}, 0$. **(38.)** $x = 4, 9, -3 \mp 4\sqrt{-10}$; $y = 9, 4, -3 \pm 4\sqrt{-10}$. **(39.)** $x = \frac{1}{29}(15 \pm 6\sqrt{-1}), 5, 1$; $y = \frac{1}{29}(25 \pm 10\sqrt{-1}), 3, \frac{3}{5}$. **(40.)** $x = \frac{8}{4}$; $y = 16$. **(41.)** $x = 4, 1, 0$; $y = 8, 0$. **(42.)** $x = 2744, 8$; $y = 9604, 4$.

APPLICATIONS.

(1.) 3. **(2.)** 18, \$20. **(3.)** 10 and 3 days, 120 and 36 miles. **(4.)** 12, 36. **(5.)** 14, 10. **(6.)** 6 miles an hour. **(7.)** 4 and 5. **(8.)** $\frac{1}{2}\left(\frac{\sqrt{(4m-1)p} \pm \sqrt{3p}}{\sqrt{m-1}}\right)$, $\frac{1}{2}\left(\frac{\sqrt{(4m-1)p} \mp \sqrt{3p}}{\sqrt{m-1}}\right)$. **(9.)** $\frac{1}{2}\sqrt{5}$, $\frac{1}{4}(5 + \sqrt{5})$. **(10.)** 1, 3, 5, 7. **(11.)** 2, 3, 4, 5, 6. **(12.)** 3, 6, 12. **(13.)** 2, 4, 8. **(14.)** 5, 10, 20, 40. **(15.)** 2, 4, 8. **(16.)** 6, 8, 10, 12. **(17.)** 1, 2, 4, 8. **(18.)** 108, 144, 192, 256. **(19.)** 72, 63, 56. **(20.)** 7, 3. **(21.)** 25. **(22.)** \$960, \$1120. **(23.)** 248. **(24.)** 6 and 7 per cent. **(25.)** 3 and 14.

INEQUALITIES.

(PAGE 150, ***134.***)

(8.) $\frac{12}{25}$ and $\frac{32}{9}$. **(9.)** Any number between 15 and 20.

PART III.

DIFFERENTIATION.

(PAGE 157, ***156.***)

(3.) $15bx^2dx - 60xdx + 4dx$. **(4.)** $2Axdx + 3Bx^2dx + 4Cx^3dx$.

(7 to 12.) $\frac{4xydx-2x^2dy}{3y^2}$; $\frac{4xdx}{(x^2+1)^2}$; $2x^{\frac{1}{2}}zdz + \frac{z^2dx}{2\sqrt{x}}$; $3x^2y^2dy+2xy^3dx+6dx$; $(5x^4-12x^3+12x^2-2x)dx$; $(x-2x^2+1)dx$. **(13 to 17.)** $15(a^3+x^3)^4x^2dx$; $\frac{9}{2}(3x-2)^{\frac{1}{2}}dx$; $4(2-x^2)^{-3}xdx$; $-\frac{dx}{2(1+x)^{\frac{3}{2}}}$. **(18 to 22.)** $-\frac{dx}{(1+x)^2}$; $-\frac{2dx}{(1+x)^3}$; $-\frac{3dx}{(1+x)^4}$; $\frac{5mdx}{(1+x)^6}$; $\frac{3mdx}{(1+x)^4}$. **(23.)** When $x > 1$, faster; also faster when $x < \frac{1}{2}$ and $> \frac{1}{3\sqrt{2}}$. When $x = \frac{1}{3\sqrt{2}}$, they both change at the same rate. When $x < \frac{1}{3\sqrt{2}}$, y changes slower than x.

INDETERMINATE COEFFICIENTS.—DEVELOPMENT OF FUNCTIONS.

(PAGE 161, ***161.***)

(3.) $1-\frac{1}{2}x^2-\frac{1}{8}x^4-\text{etc.}$; $x-x^2+x^3-x^4+\text{etc.}$; $\frac{d}{b}+\frac{ad}{b^2}x+\frac{a^2d}{b^3}x^2+\frac{a^3d}{b^4}x^3+\text{etc.}$; $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4+\text{etc.}$ **(4.)** $2+2x-3x^2-5x^3-x^4-\text{etc.}$; $1+2x+3x^2+6x^3+9x^4+\text{etc.}$; $\frac{1}{2x^2}-\frac{5}{4x}+\frac{15}{8}-\frac{45}{16}x+\frac{135}{32}x^2-\text{etc.}$ **(5.)** $1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3-\text{etc.}$; $1+\frac{2}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3-\text{etc.}$

DECOMPOSITION OF FRACTIONS.

(PAGE 164, ***167.***)

(2 to 6.) $\frac{5}{3(x-2)}-\frac{2}{3(x+1)}$; $\frac{3}{2(x-2)}-\frac{1}{2x}$; $\frac{5}{x-4}-\frac{4}{x-3}$; $\frac{7}{2(x-4)}-\frac{1}{2(x-2)}$; $\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}$. **(7 to 11.)** $\frac{2}{(x-1)^3}-\frac{1}{(x-1)^2}+\frac{3}{x-1}$; $\frac{2}{(x+3)^3}-\frac{3}{(x+3)^2}+\frac{1}{x+3}$; $\frac{1}{x^3}-\frac{1}{x^2}+\frac{2}{x}+\frac{1}{4(1-x)}-\frac{1}{2(1+x)^2}-\frac{7}{4(1+x)}$; $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}$; $\frac{1}{25(x-2)^2}-\frac{2}{125(x-2)}+\frac{1}{25(x+3)^2}+\frac{2}{125(x+3)}$. **(12 to 18.)** $\frac{1}{x^2+1}-\frac{2x-2}{(x^2+1)^2}$; $\frac{2}{x}+\frac{3(x+4)}{(x^2-2)^2}+\frac{1}{x^2-2}+\frac{1}{x-1}$; $\frac{1}{x^2}-\frac{2}{x}+\frac{3}{x+1}$; $\frac{1}{6}\left\{\frac{1}{x-1}-\frac{1}{x+1}+\frac{x-2}{x^2-x+1}-\frac{x+2}{x^2+x+1}\right\}$; $\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}$; $\frac{1}{(a-b)(x-a)}-\frac{1}{(a-b)(x-b)}$; $\frac{18}{x-3}-\frac{2}{x-1}-\frac{10}{x-2}$.

THE BINOMIAL FORMULA.

(PAGE 167, ***171.***)

(1 to 6.) $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$; $x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$; $a^n-na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2-\frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3+\frac{n(n-1)(n-2)(n-3)}{\underline{|4}}a^{n-4}x^4-\text{etc.}$; $1+4x+6x^2+4x^3+x^4$; $1-5y+10y^2-10y^3+5y^4-y^5$; $1-ny+\frac{n(n-1)}{2}y^2-\frac{n(n-1)(n-2)}{\underline{|3}}y^3+\frac{n(n-1)(n-2)(n-3)}{\underline{|4}}y^4+\text{etc.}$ **(7 to 11.)** * * * * *. **(13, 14 to 17.)** * * * *. **(18 to 20.)** $a^{\frac{2}{3}}-\frac{1}{3}a^{-\frac{4}{3}}x^2-\frac{1}{9}a^{-\frac{10}{3}}x^4-\frac{5}{81}a^{-\frac{16}{3}}x^6-\text{etc.}$; $\frac{1}{27a^3}+\frac{x^2}{27a^4}+\frac{2x^4}{81a^5}$

$+\frac{10x^6}{729a^6}+$ etc.; $a^8+4a^6c^{\frac{1}{2}}+6a^4c+4a^2c^{\frac{3}{2}}+c^2$. **(23.)** $-\frac{20}{6561}a^{-\frac{32}{3}}b^{12}$, $-\frac{10868}{1594323}a^{-\frac{50}{3}}b^{18}$.

LOGARITHMS.

(PAGE 179, ***199.***)

(1.) 4, 6, * * *. **(2.)** -2, * * *. **(3.)** 4, * *. **(5.)** To the 3291147th power, and the 1000000th root extracted. **(6.)** The 1000000th root of the 3414639th power. **(7 to 9.)** * * * *. **(10.)** .23108, .17677, * *. **(11.)** $\bar{4}.449419$, $\bar{4}.627084$, $\bar{1}.890210$. **(12.)** 12.42, .00010031, 18.3625, 1.8358. **(14.)** $\frac{2}{3}\log x+\frac{1}{2}[\log(1+x)+\log(1-x)]$, $\frac{1}{2}(\log a+\log x-\log b-\log y)$, $\frac{1}{2}[\log(s-a)+\log(s-b)+\log(s-c)-\log s]$, $\frac{1}{3}[\log x+\log(1-x)]-\frac{3}{2}\log y$; $\frac{1}{n}(m\log a+p\log b-t\log c)$, $\frac{1}{2}\log c-\frac{1}{t}\log d+\frac{1}{n}[\log(m+x)+\log(m-x)]-m\log a+n\log b$. **(16.)** $-\frac{dx}{1-x}$, $\frac{mdx}{x}$, $\frac{3mdx}{x}$, $-\frac{mdx}{x}$, $\frac{mdx}{2(1+x)}$. **(19.)** .665712, 4.49134.

SUCCESSIVE DIFFERENTIATION.

(PAGE 182, ***204.***)

(2.) $12xdx^2$. **(5.)** * * *. **(7.)** $2[(x-b)+(x-c)+(x-a)]dx^2$.

DIFFERENTIAL COEFFICIENTS.

(PAGE 185, ***207.***)

(6.) $10x^4+12x^3-10x$, $40x^3+36x^2-10$, $120x^2+72x$, $240x+72$, 240; * * * *.

TAYLOR'S FORMULA.

(PAGE 188, ***212.***)

(2.) * * *, $x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}y-\frac{1}{8}x^{-\frac{3}{2}}y^2+\frac{1}{16}x^{-\frac{5}{2}}y^3-\frac{5}{128}x^{-\frac{7}{2}}y^4+\frac{7}{256}x^{-\frac{9}{2}}y^5-$ etc.; $x^{-2}+2x^{-3}y+3x^{-4}y^2+4x^{-5}y^3+5x^{-6}y^4+6x^{-7}y^5+$ etc.; $x^{-\frac{2}{3}}-\frac{2}{3}x^{-\frac{5}{3}}y+\frac{5}{9}x^{-\frac{8}{3}}y^2-\frac{40}{81}x^{-\frac{11}{3}}y^3+\frac{110}{243}x^{-\frac{14}{3}}y^4-\frac{308}{729}x^{-\frac{17}{3}}y^5+$ etc. PAGE 189, **(2.)** $3x^5-2x^2+(15x^4-4x)h+(30x^3-2)h^2+30x^2h^3+15xh^4+3h^5$.

INDETERMINATE EQUATIONS.

(PAGE 192, ***218.***)

(4.) (a) $\begin{cases} y = 4, & 9, & 14, & 19. \\ x = 42, & 31, & 20, & 9. \end{cases}$ (b) None.

(c) $\begin{cases} y = 5, 14, 23, 32, 41, 50, 59, 68, 77, 86, 95, 104, 113, 122, 131, 140, 149. \\ x = 215, 202, 189, 176, 163, 150, 137, 124, 111, 98, 85, 72, 59, 46, 33, 20, 7. \end{cases}$

(d) $\begin{cases} y=23, 6. \\ x=17, 28. \end{cases}$ (e) $\begin{cases} y= 8. \\ x=20. \end{cases}$ (f) $\begin{cases} y= 9, 28, 47, \text{etc.} \\ x=56, 173, 290, \text{etc.} \end{cases}$ (g) $\begin{cases} y=2, 119, 236, \text{etc.} \\ x=3, 131, 259, \text{etc.} \end{cases}$

(h) $\begin{cases} y=13. \\ x= 7. \end{cases}$ (i) $5x+9y=40$. None. $5x-9y=40$, $\begin{cases} y= 5, 10, 15, \text{etc.} \\ x=17, 26, 35, \text{etc.} \end{cases}$

(k) $5x+9y=37$, $\begin{cases} y=3. \\ x=2. \end{cases}$ $5x-9y=37$, $\begin{cases} y= 2, 7, 12, \text{etc.} \\ x=11, 20, 29, \text{etc.} \end{cases}$

APPLICATIONS.

(2.) Yes; 15, 163, 9. (4.) No; yes, in an infinite number of ways; 4 3-shilling pieces and 192 guineas; possible; possible; possible. (5.) 190.

INDETERMINATE EQUATIONS BETWEEN THREE QUANTITIES.

(PAGE 194, ***219.***)

(2.) $\begin{cases} z= 1, 2, 3, 4, 5, 6, 11, 12, 13, 14. \\ y=11, 9, 7, 5, 3, 1, 8, 6, 4, 2. \\ x=10, 11, 12, 13, 14, 15, 1, 2, 3, 4. \end{cases}$ (3.) 59 sets of values.

(4.) $z=1$ $\begin{Bmatrix} y= 2, 4, 6, 8, 10. \\ x=15, 12, 9, 6, 3. \end{Bmatrix}$ $z=2$ $\begin{Bmatrix} y= 1, 3, 5, 7, 9. \\ x=14, 11, 8, 5, 2. \end{Bmatrix}$ $z=3$ $\begin{Bmatrix} y= 2, 4, 6, 8. \\ x=10, 7, 4, 1. \end{Bmatrix}$

$z=4$ $\begin{Bmatrix} y=1, 3, 5. \\ x=9, 6, 3. \end{Bmatrix}$ $z=5$ $\begin{Bmatrix} y=2, 4. \\ x=5, 2. \end{Bmatrix}$ $z=6$ $\begin{Bmatrix} y=1, 3. \\ x=4, 1. \end{Bmatrix}$

(PAGE 194, ***220.***)

(1.) $z=7, y=2, x=10$. (2.) $z=15, 30$, $y=82, 40$, $x=15, 50$. (3.) None.

APPLICATIONS.

(1.) 8 of 1st, 6 of 2d, 2 of 3d, and in 9 other ways; 23 and 2, 16 and 5, 9 and 8, 2 and 11. (2.) \$4, \$2, \$7; infinite variety of prices. (3.) 6, 3, 1, 16. (4.) Number of the 3d kind equals twice the number of the 1st kind, plus the number of the 2d kind; 1 of 1st, 6 of 2d, 8 of 3d kind. (5.) 40, 60, 24. (6.) 55, 10, 85 is one result in integers. There are an infinite number of other ways. (7.) * *. (8.) $z = 10, y = 1, x = 13$.

LOCI OF EQUATIONS.

A LARGE number of these constructions are exhibited in the text, and to give more would be to destroy the possibility of the student's deriving any benefit from the exercise.

HIGHER EQUATIONS.—TRANSFORMATION.

(PAGE 205, *228.*)

(2.) Multiply by y^4, and then put $y=x^{10}$. Finally put $x=\frac{z}{k}$, etc.

It is not deemed expedient to give farther explanations.

(PAGE 214, *249.*)

(2 to 34.) To give the roots of these equations would destroy the practical value of the examples.

(PAGE 216, *250.*)

(1.) $x^3-2x^2-11x+12=0$. (2.) $x^4-2x^3-5x^2+4x+6=0$. (3.) * * *.
(4.) $x^3-x^2-7x+15=0$. (5.) * * *. (6.) $30x^3-17x^2-11x+6=0$.
(7 to 10.) * * * * * *. (11.) $x^6-10x^5+33x^4-56x^3-73x^2+66x+39=0$.

EQUATIONS WITH INCOMMENSURABLE ROOTS.

(PAGES 216–247.)

To give the answers to these examples would be to destroy their value to the student.

CARDAN'S PROCESS.

(PAGE 251, *281.*)

(2.) $-1, 2, 2$. (3.) $2, -1 \pm \sqrt{3}$. (4.) $\sqrt[3]{4}-\sqrt[3]{2}$, and the roots of $x^2+\left(\sqrt[3]{4}-\sqrt[3]{2}\right)x+\left(\sqrt[3]{4}-\sqrt[3]{2}\right)^2+6=0$, which are $-\frac{1}{2}\left(\sqrt[3]{4}-\sqrt[3]{2}\right)$ $\pm\sqrt{-\frac{3}{2}\sqrt[3]{2}-\frac{3}{4}\sqrt[3]{4}-3}$. (5.) $\left(a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)^3$, $\left\{-\frac{1}{2}\left(a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)\pm\frac{1}{2}\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}\right)\sqrt{-3}\right\}^3$.
(6.) $1, -2\pm 3\sqrt{-1}$. (7.) * * *. (8.) $2, 2\pm\sqrt{-1}$. (9.) $8, -4, -4$.
(10.) * * *. (11.) * * *. (12.) * * *. (13.) One root is 2.32748+
(14.) $\frac{ac^{\frac{2}{3}}}{a^{\frac{2}{3}}-c^{\frac{2}{3}}}$.

DESCARTES'S PROCESS.

(PAGE 252, *283.*)

Ex. 4, -2, $-1+\sqrt{-1}$, and $-1-\sqrt{-1}$.

RECURRING EQUATIONS.

(PAGE 255, *291.*)

(1.) $2\pm\sqrt{3}$, $\frac{1}{2}(1\pm\sqrt{-3})$. (2.) -1, $\frac{1}{2}(9\pm\sqrt{77})$, $\frac{1}{2}(3\pm\sqrt{5})$. (3.) 1, 3, $\frac{1}{3}$, -2, $-\frac{1}{2}$. (4.) -1, $\frac{1}{2}m\pm\sqrt{\frac{1}{4}m^2-1}$, in which $m=\dfrac{1+4a\pm\sqrt{5+20a}}{2(1-a)}$. (5.) -1, 1, 1, -1, -1, $\frac{1}{2}(1\pm\sqrt{-3})$. (6.) 2, $\frac{1}{2}$, $\frac{1}{2}m\pm\sqrt{\frac{1}{4}m^2-1}$, in which $m=\frac{1}{4}(-5\pm\sqrt{5})$. (7.) 2, $\frac{1}{2}$, 2, $\frac{1}{2}$, $\frac{1}{2}(1\pm\sqrt{-3}$. (8.) $\dfrac{a}{2}(3\pm\sqrt{5})$, $\dfrac{a}{2}(-7\pm3\sqrt{5})$. (9.) $\frac{1}{4}(\sqrt{5}-1\pm\sqrt{-10-2\sqrt{5}})$, $-\frac{1}{4}(\sqrt{5}+1\mp\sqrt{-10+2\sqrt{5}})$. (10.) $\frac{1}{2}m\pm\sqrt{\frac{1}{4}m^2-1}$, in which $m=2(1\pm\sqrt{3})$.

BINOMIAL EQUATIONS.

(PAGE 255, *292.*)

(**1** to **5**.) See answer to (**45**), page 138, and multiply them respectively by $\sqrt[3]{5}$, $\sqrt[4]{3}$, $\sqrt[5]{2}$, $\sqrt[6]{7}$, $\sqrt[8]{11}$. (**7**.) See as above.

EXPONENTIAL EQUATIONS.

(PAGE 256, *206.*)

(**2** to **6**.) 3.0957+, 11.384+, 3.292+, 0, 0. (**8**.) 3.597+. (**9**.) 2.316+. (**10**.) 2.879+. (**11**.) 3.233+. (**12**.) 2.001+. (**13**.) $\dfrac{\log(b\pm\sqrt{b^2+c})}{\log a}$. (**14**.) $\dfrac{2\log a+\log c}{b\log a}$. (**15**.) $\dfrac{\log c}{m\log a+n\log b}$. (**16**.) $2\frac{1}{4}$, $3\frac{3}{8}$. (**17**.) $\left(\dfrac{a}{b}\right)^{\frac{b}{a-b}}$, $\left(\dfrac{a}{b}\right)^{\frac{a}{a-b}}$. (**18**.) $\frac{1}{2}\left(q+\dfrac{\log n}{\log m}\right)$, $\frac{1}{2}\left(q-\dfrac{\log n}{\log m}\right)$. (**19**.) $\dfrac{3(2+\log 5)}{2\log 2+3\log 3}$, $\dfrac{2(2+\log 5)}{2\log 2+3\log 3}$. (**20**.) 2.421, -1.421. (**21**.) $\dfrac{\log(a-b)}{\log(a+b)}$. (**24**.) \$196.71, \$198.98, \$200.17, \$259.37, \$265.33, \$268.51, \$180.61, \$181.43, \$134.83.

(25.) 7.13, 10.24, 16.23, 20.48 years. **(26.)** 29.91 years. **(27.)** $1502.63. **(28.)** $1933.97. **(30.)** $1157.28. **(32.)** $4794.52, $3500. **(33.)** $577.06. **(35.)** The former by $629.03. **(36.)** $500.91. **(38.)** 13.58 years. **(39.)** $796.87. **(40.)** $8229.70. **(41.)** Gains $1756.60.

APPENDIX.

SERIES.

(Page 275, ***311.***)

(2.) 12, 6, 0. **(3.)** 8, 32. **(4.)** $-\frac{1}{32}$. **(5.)** 1. **(6.)** -14.

(Page 276, ***313.***)

(4.) $-x^3$, $+x^2$, $+x$. $5x^8$, $5x^9$. **(5.)** -27, $+9$, $+3$. 32805, 98415. **(6.)** $+3x^2$, $+2x$. $1093x^7$, $3281x^8$. **(7.)** -4, $+4$. 192, 448. **(8.)** $-2x^3$, $+x^2$, $+2x$. $87x^5$, $173x^6$. **(9.)** $-\frac{c}{b}x$. $\frac{ac^4}{b^5}x^4$, $-\frac{ac^5}{b^6}x^5$. **(10.)** -1, $+4$, -6, $+4$. 56, 84, 120. **(11.)** $+1$, -3, $+3$. 26, 34, 43.

(Page 277, ***314.***)

(1.) $4516x^7$. **(2.)** $17x^8$. **(3.)** $-x^9$. **(4.)** 2733. **(5.)** 29525. **(6.)** 1365. **(7.)** 20. $\frac{n(n+1)}{2}$. **(8.)** $n(n+1)$. **(9.)** $6396x^{11}$. **(10.)** $+1$, -5, $+10$, -10, $+5$. $+1$, -3, $+3$. $+1$, -3, $+3$. $+3x^3$, $-x^2$, $+2x$. **(11.)** n^2. **(12.)** 8694. **(13.)** 26, 34, 43, 53, 64. $26x^{12}$, $34x^{14}$, $43x^{16}$, $53x^{18}$, $64x^{20}$. 196, 336, 540, 825, 1210, 1716.

(Page 279, ***315.***)

(2.) No. **(3 to 6.)** Yes.

(Page 282, ***316.***)

(1.) $\frac{1-x}{1-3x-2x^2}$. **(2.)** $\frac{1+x}{1-x-x^2}$. **(3.)** $\frac{1+x}{(1-x)^2}$. **(4.)** $\frac{3-x-6x^2}{1-2x-x^2+2x^3}$. **(5.)** ∞. **(6.)** $\frac{1}{1-x-x^2+x^3}$. **(7.)** $\frac{a}{b+cx}$. **(8.)** 400; n^2. **(9.)** 1275; $\frac{n(n+1)}{2}$. **(10.)** 278256; $\frac{n^5+10n^4+35n^3+50n^2+24n}{120}$. **(11.)** 1031550; $\frac{6n^4+44n^3+99n^2+61n}{3}$. **(12.)** 60710; $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$.

(13.) $\frac{n(n+1)(2n+1)}{6}$. (14.) $\frac{n^2(n+1)^2}{4}$. (16.) $\frac{1}{2}$. (18.) $\frac{11}{18}$. $\frac{1}{3}\left(\frac{11}{6}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right)$. (20.) $\frac{3}{4}$. (21.) $\frac{1}{4}$. (22.) $\frac{1}{12}$. (23.) 1. (25.) $\frac{13}{240}$. (26.) $\frac{1}{8}$. (27.) $\frac{5}{24}$. (28.) $\frac{1}{18}$. (29.) $\frac{1}{72}$. (30.) $\frac{7}{2916}$.

PILING BALLS AND SHELLS.

(PAGE 287, *322.*)

(1.) 1540, 13244, 903. (2.) 9455, 4324, 35720, 465, 276, 1128. (3.) 7490, 3880. (4.) 624. (5.) 2730. (6.) 36256.

REVERSION OF SERIES.

(PAGE 288, *323.*)

(2.) $x=y-y^2+y^3-y^4+$ etc. (3.) $x=y-3y^2+13y^3-67y^4+$ etc. (4.) $x=y+\frac{1}{3}y^3+\frac{2}{15}y^5+\frac{17}{315}y^7+$ etc. (5.) $x=\frac{1}{2}y-\frac{3}{16}y^3+\frac{19}{128}y^5-$ etc. (6.) $x=(y-1)-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-\frac{1}{4}(y-1)^4+$ etc. (7.) $x=\frac{y}{m}+\frac{(am^2-n)y^2}{m^3}+\frac{[bm^4-mp-2n(am^2-n)]y^3}{m^5}+$ etc.

INTERPOLATION.

(PAGE 290, *325.*)

(2.) 1.794. (5.) (* * *).

PERMUTATIONS.

(PAGE 293, *334.*)

(1.) 720, 24, 3,628,800. (2.) 720, 42, 210, 840, 2520, 5040, 5040. 120, 21, 35, 35, 21, 7, 1. (3.) 36, 84, 126, 126, 84, 36, 1. (4.) 72, 504, 3024. (5.) 20. (6.) 35, 21. (7.) 127. (8.) 479,001,600. (9.) 792. (10.) 15. (11.) 166,320. 64,864,800. (12.) 1023.

PROBABILITIES.

(PAGE 295, *335.*)

(1.) $\frac{59}{60}$, $\frac{1}{60}$; $\frac{39}{40}$, $\frac{1}{40}$. (2.) $\frac{2}{9}$, $\frac{5}{6}$, 2:1, 4:3. (4.) 6 to 1. (5.) (* * *). (6.) $\frac{1}{6}$, 5 to 1. (11.) \$$2\frac{18}{31}$.

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